SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER USAFA-TR-80-6	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) An Analysis of Air Force Economic Order Quantity Type Inventory Data with an Application to Reorder Point Calculation		5. TYPE OF REPORT & PERIOD COVERED Final Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Charles R. Mitchell, LtCol, USAF Robert A. Rappold, LtCol, USAF Wayne B. Faulkner (Formerly of AFIMC)		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Dept of Mathematical Sciences US Air Force Academy, CO 80840		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Logistics Management Center Gunter AFS, AL		12. REPORT DATE Apr 1980
		13. NUMBER OF PAGES 64
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report)
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Logistics, Reorder Point, Demand during lead time, Poisson, Geometric, Compound distribution

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

One of the important uses of an EOQ item's distribution of lead time demand is to set its reorder point (when to order new stock). The current Air Force system computes the reorder point by assuming that demand during lead time is normally distributed. The analysis presented here shows that a much more realistic model of observed demand patterns can be chosen from the compound Poisson family of distributions. The geometric-Poisson and constant-Poisson members of that family are used in this study. Also we allow for fixed and

DD FORM 1473

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. Abstract (continued)

variable lead times. The ultimate objective throughout the study is to under-
stand, as exactly as possible, the true underlying random processes involved
in the EOQ supply system. A strong secondary objective, though, is to allow
for a probability model that could be implemented in the environment of a
large base supply account.

Data from eight federal stock groups and four bases (three CONUS and one
overseas) are analyzed. Approximately 10,000 items are involved. We analyze
individual customer arrival and demand processes as well as empirical lead
time data. The primary emphasis is on fitting the assumed geometric-Poisson
and constant-Poisson models to the data. The geometric-Poisson model is taken
to be a better representation of the actual arrival and demand processes than
the constant-Poisson model. Although the latter model is a cruder approxima-
tion, it does represent an easier model for possible implementation. Both
models are shown to be more representative of the actual data than is the
normal distribution assumption. Also we show that both models perform better
in setting hypothetical reorder points than the current model.

Only a cursory examination of the effect of variable lead time is attempted
since the data contain many outliers. The data are sufficient, though, for
testing the sensitivity of the reorder point computation to variable lead time.
Our analyses show the effect to be almost neglible for CONUS base - CONUS
resupplier combinations. The effect for overseas base - CONUS resupplier
combinations probably is significant.

FORTRAN programs are available for all of the analyses described.

AN ANALYSIS OF AIR FORCE ECONOMIC ORDER QUANTITY TYPE
INVENTORY DATA WITH AN APPLICATION TO REORDER POINT CALCULATION

by

LtCol C.R. Mitchell
LtCol R.A. Rappold
Dept of Mathematical Sciences
USAF Academy, CO

Mr W.B. Faulkner
formerly of the
Air Force Logistics Management Center
Gunter AFS, AL

April 1980

PREFACE

The research reported here was for the Air Force Logistics Management Center at Gunter AFS, AL. Of primary interest is the distribution of demand during lead time for economic order quantity (EOQ) type items. Not only does this research support an implementation plan for the new DOD Instruction 4140.45 but it contributes to the basic understanding of demand processes for EOQ items.

SUMMARY

One of the important uses of an EOQ item's distribution of lead time
demand is to set its reorder point (when to order new stock). The
current Air Force system computes the reorder point by assuming that
demand during lead time is normally distributed. The analysis presented
here shows that a much more realistic model of observed demand patterns
can be chosen from the compound Poisson family of distributions. The
geometric-Poisson and constant-Poisson members of that family are used
in this study. Also we allow for fixed and variable lead times. The
ultimate objective throughout the study is to understand, as exactly
as possible, the true underlying random processes involved in the EOQ
supply system. A strong secondary objective, though, is to allow for a
probability model that could be implemented in the environment of a
large base supply account.

Data from eight federal stock groups and four bases (three CONUS and
one overseas) are analyzed. Approximately 10,000 items are involved.
We analyze individual customer arrival and demand processes as well as
empirical lead time data. The primary emphasis is on fitting the
assumed geometric-Poisson and constant-Poisson models to the data. The
geometric-Poisson model is taken to be a better representation of the
actual arrival and demand processes than the constant-Poisson model.
Although the latter model is a cruder approximation it does represent
an easier model for possible implementation. Both models are shown to
be more representative of the actual data than is the normal distribution
assumption. Also we show that both models perform better in setting
hypothetical reorder points than the current model.

Only a cursory examination of the effect of variable lead time is
attempted since the data contain many outliers. The data are sufficient,
though, for testing the sensitivity of the reorder point computation to
variable lead time. Our analyses show the effect to be almost neglible
for CONUS base - CONUS resupplier combinations. The effect for overseas
base-CONUS resupplier combinations probably is significant.

FORTRAN programs are available for all of the analyses described.

ii

TABLE OF CONTENTS

LIST OF FIGURES

SECTION 1

INTRODUCTION AND BACKGROUND

Several probability models exist to describe the total units demanded for an economic order quantity (EOQ) type inventory item during stock replenishment (lead time). In this report both demand and lead time are assumed to be random variables and we empirically study their probability distributions. Data from several typical Air Force base supply accounts are used.

The research reported here is in support of an implementation plan for DOD Instruction 4140.45 [4]. This instruction pertains to a standard stockage policy for consumable secondary items at the so-called retail level of inventory. An integral part of the DODI is the specification of a probability model for the distribution of an item's lead time demand.

The distribution of lead time demand can be used in an important application in inventory models. This application is in the formation of an item's total variable cost per year which involves the ordering, holding and backorder costs (Appendix A gives the details). The decision variables in this formulation are the reorder point (when to order) and the reorder quantity (how much to order). An exact solution for the decision variables involves complex nonlinear equations. An approximate solution involves separating the total variable cost in such a way that the reorder quantity is given by the well known Wilson EOQ formula and the reorder point is the mean demand during lead time plus some number of standard deviations of lead time demand. The quantity added to the mean lead time demand is commonly referred to as a safety level. Equations A-13 and A-16 in the appendix give the approximate reorder quantity and reorder point, respectively.

The Air Force currently uses the above approximation method. In calculating the reorder point for base supply accounts in the CONUS the safety level is taken to be one standard deviation of lead time demand, where lead time demand is assumed to be normally distributed. This

1

calculation implies a 0.84 probability of satisfying all demand during lead time. In estimating the standard deviation, the Air Force further assumes that the variance-to-mean ratio of lead time demand is three for every item. We show later that the normal distribution does not provide a reasonable fit to the lead time demand nor is a variance-to-mean ratio of three realistic. A more appropriate manner of computing the reorder point is the subject of this research.

Our primary objective is to adequately describe the empirical probability distribution of lead time demand. A reasonable model of lead time demand can be used to determine a reorder point directly from an equation similar to A-15 in Appendix A (the quantity L, lead time, in that equation is assumed to be fixed; we want to allow variable lead times as well). A strong secondary objective is to determine a reasonable probability model that is implementable in the environment of a typical large base supply account.

The remainder of this paper is organized as follows: Section 2 describes the assumed probability models, Section 3 shows the available data, Section 4 gives empirical evidence to support the probability models, Section 5 discusses the effects of variable lead time, and Section 6 contains our recommendations for validation/implementation.

SECTION 2
PROBABILITY MODELS FOR LEAD TIME DEMAND

It has been stated that the normal distribution is not a reasonable model for the distribution of an item's lead time demand. The credibility of this statement can be determined from Figure 1 where we show the demand histories of several typical items. Shown for each item is the date of customer demand and the units per demand. For example, item 1 had a customer demand of 1 unit on day 122 of 1977. Since lead time demand is the sum of several customer's units per demand the normal distribution is often used to represent demand, particularly when a large number of customers is expected to arrive during lead time (the central limit theorem [5] for sums of random variables is the justification). For the kinds of data shown in Figure 1 the number of customer arrivals during lead time is usually very small. For one of our base supply accounts, which we feel is representative, 94% of about 1300 items had zero or one customer arrivals during typical lead times. In no reasonable way can the central limit theorem apply to random sums associated with so few arrivals, particularly when the units per demand are of the magnitude shown in Figure 1. It is natural, therefore, to search for more applicable probability distributions for lead time demand.

Item #1		Item #2		Item #3		Item #4	
Date	Units	Date	Units	Date	Units	Date	Units
77-122	1	77-307	1	77-134	6	77-265	3
77-152	1	77-313	1	77-171	2	78-044	3
77-154	1	77-327	1	77-186	5	78-052	3
77-165	1	77-332	1	77-249	1		
77-230	1			78-073	2		
77-242	2						
77-342	1						
77-347	1						

Figure 1. Typical Demand Histories

3

Our objective is to identify probability models which adequately describe customer arrival and demand patterns like those in Figure 1. Three probability models of lead time demand are presented in this section—the first two deal with constant lead time and the third deals with stochastic lead time.

Model I, Geometric-Poisson

Suppose that during a constant lead time of length ℓ, there are $N(\ell)$ customers who request a certain item. If the demands per customer are represented by U_i, $i = 1,2,\ldots, N(\ell)$, then the demand during time ℓ is

$$D(\ell) = U_1 + U_2 + \ldots + U_{N(\ell)}. \tag{1}$$

In (1), $N(\ell)$ and the U_i's are random variables; thus, by construction $D(\ell)$ is also a random variable.

Figure 2 gives a graphical display of this process. Shown in the figure are demands by four customers ($N(\ell) = 4$); they request two, one, one and three units, respectively ($U_1=2$, $U_2=1$, $U_3=1$, $U_4=3$). The lead time demand is $D(\ell) = 7$. This process matches the kinds of arrival and demand patterns that were shown in Figure 1.

Units
demanded
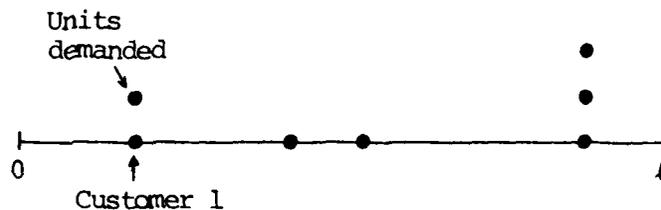
Figure 2. Example of Lead Time Demand Process

For a particular item we assume that the U_i's are independent and identically distributed. We choose the probability distribution of U (we drop the subscript) to be the geometric distribution. As for the distribution of the number of customers who demand the item in ℓ units, $N(\ell)$, we assume the Poisson distribution. Numerous articles in the inventory literature assume these models [2,6,16,17,18,19]. However, no

4

data to support these models are given in any of the references. A
later section of this report shows how these models fit actual Air Force
inventory data. Table 1 shows the form of the geometric and Poisson
probability distributions and their means and variances.

Table 1. Probability Distributions

Name	Probability Distribution	Mean	Variance
Geometric	$\Pr(U=u) = (1-p)p^{u-1}$, $u=1,2,\ldots$	$1/(1-p)$	$p/(1-p)^2$
Poisson	$\Pr(N=n) = (\lambda\ell)^n e^{-\lambda\ell}/n!$, $n=0,1,2,\ldots$	$\lambda\ell$	$\lambda\ell$
Geometric-Poisson	$P_0 = e^{-\lambda\ell}$ $$P_x = \frac{(1-p)\lambda\ell}{x} \sum_{j=1}^{x} jp^{j-1}P_{x-j}, x\geq 1$$ where $P_x = \Pr(D=x)$	$\lambda\ell/(1-p)$	$\lambda\ell(1+p)/(1-p)^2$
Constant-Poisson	$P_x = (\lambda\ell)^{x/c} e^{-\lambda\ell}/(x/c)!$, $x=0,c,2c,\ldots$ where $P_x = \Pr(D=x)$	$c\lambda\ell$	$c^2\lambda\ell$

With the assumption of Poisson customer arrivals the distribution
of lead time demand in (1) is called a compound Poisson distribution
[18]. The geometric assumption for the units demanded per customer
gives rise to a distribution called geometric-Poisson [18]. The form
of this distribution and its mean and variance are shown in Table 1.
Table 2 shows a portion of the probabilities for the geometric-Poisson
distribution for several values of the mean (m) and for an arbitrary
variance-to-mean ratio (VMR).

The geometric-Poisson distribution can be used in a straightforward
way to compute a reorder point. Suppose that the mean lead time demand
for an item is five and the variance-to-mean ratio of lead time demand
is known to be 2.5. If we further assume that the probability distri-
bution of lead time demand is the geometric-Poisson distribution, then
the probabilities in Table 2 (with m = 5) are applicable. For example,

Table 2. Geometric-Poisson Probabilities
For VMR = 2.5

			Probability of x demands				
$x^{\diagdown m}$	.50	1.0	2.0	3.0	4.0	5.0	6.0
0	.7515	.5647	.3189	.1801	.1017	.0574	.0324
1	.1227	.1844	.2083	.1764	.1328	.0938	.0635
2	.0626	.109?	.1573	.1620	.1437	.1167	.0895
3	.0317	.0630	.1113	.1347	.1365	.1245	.1057
4	.0159	.0356	.0753	.1047	.1191	.1201	.1115
5	.0079	.0198	.0492	.0775	.0978	.1079	.1085
6	.0039	.0109	.0313	.0551	.0765	.0917	.0993
7	.0019	.0059	.0194	.0381	.0577	.0746	.0865
8	.0010	.0032	.0119	.0256	.0421	.0585	.0724
9	.0005	.0017	.0071	.0169	.0300	.0446	.0585
10	.0002	.0009	.0042	.0109	.0209	.0331	.0460

we see that the probability of zero, one, and two demands is .0574, .0938, and .1167, respectively. We can form the cumulative distribution of lead time demand by summing the individual probabilities. That is, the probability of lead time demand being one or less is .1512, two or less is .2679, etc. Continuing in this way we see that the probability is .8452 that lead time demand will be eight units or less. If we desire a reorder point to satisfy 84% of lead time demand, then for this example the reorder point would be set to eight units. Reorder points for any other probability levels can be determined in a similar way.

A few comments about a special case of the geometric-Poisson distribution are appropriate before we leave this model. It is important to note that if p = 0 in the geometric distribution, then with probability one each customer demands exactly one unit. Lead time demand, $D(\ell)$ in equation (1), in this case corresponds to $N(\ell)$. In other words, for this special case, lead time demand is given by the Poisson distribution.

Item 2 in Figure 1 has this characteristic; it will be shown later that
many items in a typical supply account also have this property. A
model which is a particular generalization of this special case will be
presented next.

Model II, Constant-Poisson

Another compound Poisson distribution of interest here is one we
call the constant-Poisson. During a fixed lead time ℓ, suppose that
demand is given by

$$D(\ell) = \overbrace{c + c + \ldots + c}^{N(\ell) \text{ terms}} \tag{2}$$

where $N(\ell)$ customers demand c units each. With the assumption that
$N(\ell)$ has a Poisson distribution, $D(\ell)$ is again a compound Poisson distri-
bution [18]. Table 1 shows the form of the distribution and its mean
and variance.

Although this model is not as realistic for all items as the geometric-
Poisson model, it does describe demand on items like number two and four
in Figure 1. We show later that this model also describes lead time
demand reasonably well on items like number one in Figure 1 where the
variance of units per demand is very small.

Reorder points can be computed exactly as in Model I. That is,
from the cumulative probability distribution a reorder point is chosen
to satisfy a certain proportion of lead time demand.

Models I and II apply to fixed lead times; in the next model we
address stochastic lead time.

Model III, Compound Poisson with Stochastic Lead Time

In this model we allow the lead time for resupply to be a random
variable. If L denotes lead time and $N(L)$ represents the number of
customer arrivals during L, then lead time demand can be written as

$$D(L) = U_1 + U_2 + \ldots + U_{N(L)}. \tag{3}$$

In (3) the U_i's are assumed to be either geometrically distributed as in Model I or constant as in Model II. For a particular value of the lead time we assume that customer arrivals are Poisson distributed. With these assumptions we call this model a compound Poisson with stochastic lead time.

The distribution of lead time demand can be developed in the following way. Let ℓ be a particular value of the variable lead time L. In this case we essentially have Model I or II and we can represent the probability distribution by $f(d|\ell)$. Now if L is assumed to have a discrete probability distribution $g(\ell)$, it follows that the marginal distribution of lead time demand can be written as

$$f(d) = \Sigma f(d|\ell)g(\ell). \tag{4}$$

The summation is taken over all permissible values of L. (Later it will be obvious why we assume L is discrete.)

Table 3 shows an example of this process. Suppose lead time L has the three values 10, 20, and 30 days with probabilities .3, .5, and .2, respectively. If we arbitrarily use the geometric-Poisson model, then the conditional distribution of lead time demand, $f(d|\ell)$, results. The marginal distribution of lead time demand, $f(d)$, is shown in the last row. For example, $f(0) = .333(.3) + .111(.5) + .037(.2) = .163$. The other values of $f(d)$ are similarly obtained. Reorder points for any desired probability can be determined from $f(d)$ by forming the cumulative distribution function as in Model I.

Table 3. Example of Model III

ℓ	$g(\ell)$	d and $f(d|\ell)$					
		0	1	2	3	4	≥ 5
10	.3	.333	.333	.196	.089	.033	.016
20	.5	.111	.221	.242	.190	.120	.116
30	.2	.037	.110	.176	.197	.174	.306
	$f(d)$	.163	.232	.215	.161	.105	.124

A closed form representation for f(d) with the often assumed gamma distribution of lead time has proved elusive. We suspect that the probability distribution of lead time demand is intractable for the geometric-Poisson and gamma assumption. This is not a serious problem, however, since the computation for f(d), as illustrated in Table 3, can be performed easily on a computer.

Despite the above mentioned complexity of the probability distribution of D(L) in (3), its mean and variance are easy to derive. McFadden [12] shows the mean and variance to be

$$E(D(L)) = E(U)E(N)E(L) \tag{5}$$

and

$$V(D(L)) = E(L)[E(N)V(U)+V(N)E^2(U)] + V(L)E^2(U)E^2(N). \tag{6}$$

In (5) and (6), $E(\cdot)$ is expectation or mean and $V(\cdot)$ is variance. Besides lead time L, the other random variables are U (units per demand) and N (number of customer arrivals per unit time).

Now that several lead time demand models have been described, our next task is to fit them to data; the next section describes those data.

SECTION 3
DATA

For studies of this nature and others concerning retail inventory
theory and practice, several Air Force bases are providing demand and
lead time data to the Air Force Logistics Management Center (AFLMC).
The data we report are from four of these bases. Table 4 shows the
bases and federal stock groups that comprise our data set. The descrip-
tion of a stock group is not all encompassing, but is meant to be
representative of the items in that group. The eight stock groups were
chosen somewhat arbitrarily to represent a broad spectrum of typical
items. The amount of usable data varies from 6 to 12 months across the
individual stock accounts with the time frame being 1977-1978. All EOQ
items with at least one demand during the data collection period are
considered.

Table 4. Data

Base	Federal Stock Group (FSG)	Number of Items
Bentwaters	16 - Aircraft Landing Gear	397
	59 - Electronics	2,526
Dover	15 - Aircraft Structures	712
	59 - Electronics	2,318
	66 - Flight Instruments	373
Minot	29 - Engine Miscellaneous	348
	31 - Bearings	245
Randolph	53 - Screws, Nuts, Bolts	2,877
	Total	9,796

For each item there are three data records: an item record, an
order and shipping time or lead time record, and a demand record. The
item record contains an item's federal stock number, unit cost, routing
identifier and demand value (demand value is the product of an item's
unit cost and total demands for the data collection period; it thus
represents a gross measure of an item's supply importance).

The next information given is a lead time record. It contains, for each stock replenishment request, the requisition date, receipt date, order and shipping time, and issue priority code. The demand record is similar to Figure 1 where for each customer demand, the arrival date and units per demand are given. Only recurring demands are considered.

Before showing any data related to the probability models of Section 2, we describe some pertinent characteristics of the data. These characteristics will be helpful in later discussions on probability model fitting and validation/implementation. First we describe an inspection of the raw data that was performed to determine if the data are adequate for analysis purposes. Also, we present some descriptive statistics related to item demand patterns, demand value, and lead time experience data.

Raw Data Inspection

Few data sets are complete or free from errors of various kinds and the data we have to work with are no exception. A necessary first task that precedes any data analysis or probability model fitting is to examine the data for inconsistencies and/or missing data. In Figure 3 we show a calendar time plot of the number of customer arrivals per month for all items in the Bentwaters 59 and Randolph 51 supply accounts (the latter data set will not be used again in this report; however, it is a good example to emphasize the importance for "looking" at the raw data in detail).

In the Bentwaters 59 data an inordinately large number of customers made demands on the system for the period 7262 (day 262 of 1977) to 7289. This period happens to cover the start of the fiscal year (FY) for 1978. Although unsubstantiated, we suspect that fund constraints at the end of FY 1977 caused potential customers to delay their requests until after the start of the new fiscal year when more funds were available. The downward trend just before the referenced period also supports this hypothesis. This kind of supply performance represents a nonrandom characteristic that we do not attempt to model. For

11
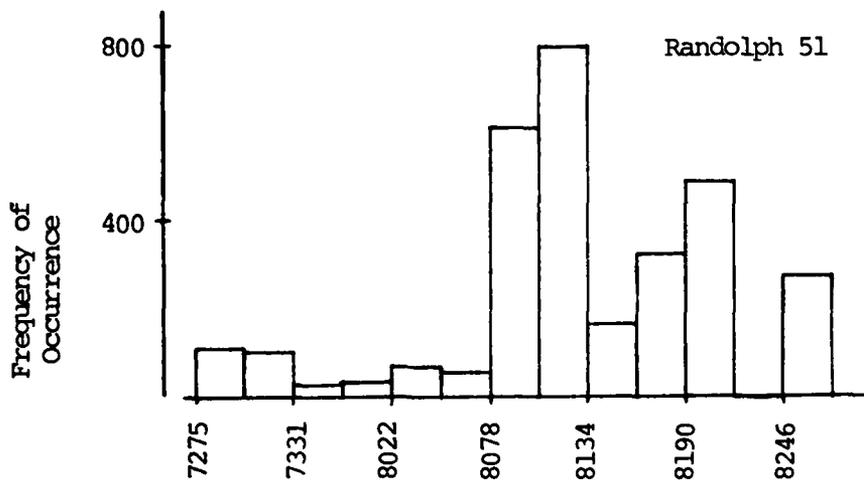
Figure 3. Calendar Plots of Customer Arrivals per Month

later analyses we delete this period of data (7262-7289) for the Bentwaters accounts.

With reference to the Randolph 51 data, several occurrences seem peculiar—one concerns the large jump in customer arrivals during the period 8078-8106, the others concern the drop for the period 8134-8162 and the total lack of data for the period 8216-8246. The problems with this account make it virtually useless for our purposes. We have no reasonable explanation of what may have caused these phenomena.

Figure 4 shows the data periods that we analyze in this study. Omissions of data are for reasons given above (these periods are labeled "omit" in the figure). For reasons that will become obvious in the next section, we choose six months of data for CONUS bases and approximately nine months of data for the single overseas base (these periods are called observation periods in the figure; the holdout periods will be used for prediction in the next section). Next we look at units demanded per customer to determine if any inconsistencies exist in those data.
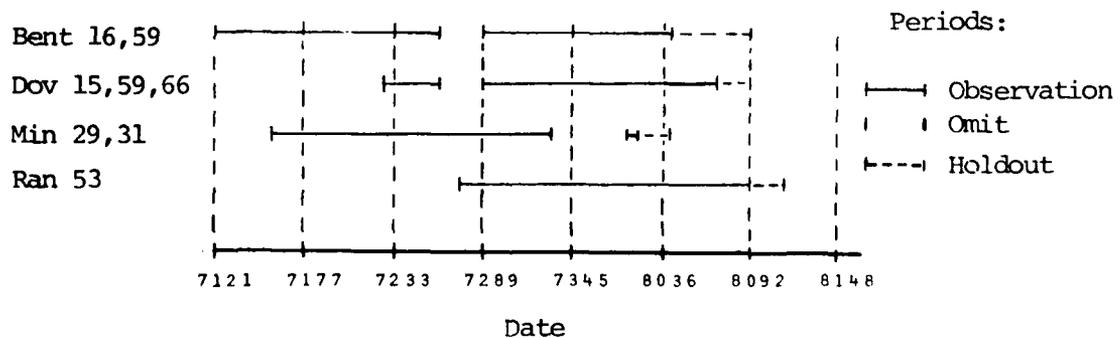
Figure 4. Data Periods for Analyses

Demand patterns like those in Figure 1 would be desirable for every item. Unfortunately, a few items have what appear to be outliers intermixed with their demand histories. For instance, one item from the Bentwaters account had units demanded per customer of 1, 2, 2, 74, and 6. The largest demand turns out to have been associated with a one time modification and should not have been included as a recurring

13

or random demand. Other items have similar data that are of a
suspicious nature—perhaps because of a planned modification or maybe
just erroneous reporting. We estimate that only 5% or so of the items
have characteristics of this sort.

It is important to emphasize again that these kinds of anomalous
data can contaminate a data analysis. In research such as this and
certainly in an operational setting where a probability model or any
estimation device is used, provisions should be included to identify
nonrandom data. We do not suggest that these unusual data should be
ignored; to the contrary, managers of the system should investigate
these occurrences in detail and attempt to preclude their happening in
the future.

Item Statistics Versus Unit Cost

The demand histories that were shown in Figure 1 were for items
that cost over $5.00. Figure 5 shows the same kind of data for a sample
of items that cost $5.00 or less. Clearly, these two figures suggest
it is characteristic of EOQ accounts that cheaper items, for the most
part, have different demand patterns than more expensive ones.

Item #1		Item #2		Item #3		Item #4	
Date	Units	Date	Units	Date	Units	Date	Units
77-123	50	77-140	4	77-175	2	77-122	12
77-144	63	77-143	6	77-178	2	77-122	5
77-151	63	77-159	6	77-265	2	77-143	14
77-159	63	77-167	1	77-265	2	77-171	5
77-193	63	77-200	1			77-192	5
77-207	63	77-272	6			77-271	5
77-237	75	77-326	4			77-277	14
77-242	63	77-361	4			78-012	15
77-265	35	78-012	4			78-044	5
77-326	63	78-086	4			78-052	15
78-003	63	78-088	4			78-072	15
78-012	63						
78-031	63						
78-038	43						

Figure 5. Typical Demand Histories (Unit Cost ≤ $5.00)

14

To investigate this further, Table 5 gives some statistics on customer arrival rates and units demanded per customer for the two strata of unit cost (UC). We see that the average number of customer arrivals per week is only slightly larger for the lower cost items. However, the dramatic difference relates to the average and variance of units demanded per customer for the two groups. The average for all items shows that lower cost items have an average units demanded per customer approximately six times as great as that for items which cost over $5.00. In addition, the variance of an item's units demanded per customer is approximately 280 times as great (comparing low cost to high cost). Not surprisingly, FSG 53 (screws, nuts, bolts) is more badly behaved than the other stock groups, at least when measured in terms of variance of units demanded per customer.

Table 5. Statistics on Customer Demands Versus Unit Cost

Base	FSG	Average No. Customers/Week		Average Units/Customer		Variance Units/Customer	
		UC≤ $5.00	UC> $5.00	UC≤ $5.00	UC> $5.00	UC≤ $5.00	UC> $5.00
Bentwaters	16	.16	.13	6.0	2.3	26.1	7.7
	59	.12	.09	4.9	2.0	210.2	3.5
Dover	15	.13	.11	4.7	1.5	12.4	.7
	59	.11	.08	8.0	1.9	1122.2	2.4
	66	.09	.09	7.1	1.7	601.2	1.3
Minot	29	.13	.09	16.6	2.2	1275.3	.7
	31	.08	.08	5.3	2.7	11.4	1.3
Randolph	53	.18	.10	21.2	3.0	1459.7	8.4
Average for All Items		.14	.10	13.0	2.1	949.6	3.4

Another way to analyze an item's demand pattern is to study the variance-to-mean ratio of its total demands over some period of time, say one week. Figure 6 shows a relative frequency chart for the VMR of total demands/week for items from the Dover 59 account. Here the median (50th percentile) is 1.4 and the 95th percentile is 6.6. It is

15

Variance-to-Mean Ratio
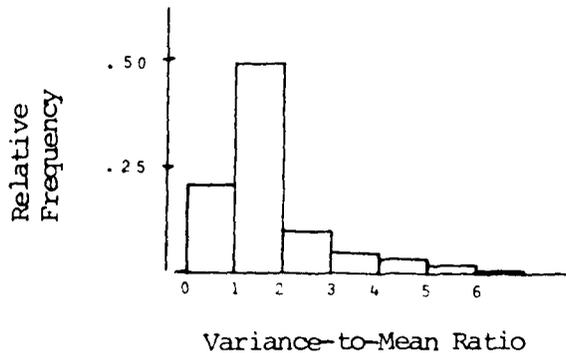
Figure 6.   Relative Frequencies of VMR for
Dover 59 (UC > $5.00)

interesting to note that on 82.3% of the items the VMR is less than
three, which is the value that the Air Force uses in computing the
safety level part of a reorder point. We have not been able to find a
report that shows the rationale for this choice of three; for these
data any constant value of the VMR applied to every item seems in-
appropriate. The large variation in the VMR across items suggests
each item's particular VMR would be a better value to use. (We do not
mean to infer that the VMR of demand over a variable lead time is the
same as the VMR over a fixed period of time. Indeed, equations (5) and
(6) can be used to show how the VMR of lead time demand varies with the
mean and variance of the lead time process. It does seem reasonable,
though, that the VMR of lead time demand would also be extremely
variable--as is true here for the VMR of demand for one week. The VMR
of lead time demand will be discussed in Section 5).

Table 6 gives the VMR of total demands/week for each of the supply
accounts in terms of the two categories of unit cost. Shown for each
account are the median and 95th percentile of the VMR. These statistics
are based on 7696 items with at least one customer arrival during the
respective observation periods shown in Figure 4 (4856 items for UC ≤
$5.00 and 2840 items for UC > $5.00). As was true for the data con-
cerning units demanded per customer, we see a large difference in the
VMR of total demands/week versus unit cost.

16

Table 6. Variance-to-Mean Ratio of Total Demands/Week

Base	FSG	UC ≤ $5.00		UC > $5.00	
		Median	95th Percentile	Median	95th Percentile
Bentwaters	16	3.7	21.0	1.5	7.5
	59	3.3	24.5	1.5	5.2
Dover	15	2.0	19.1	1.3	4.3
	59	2.9	23.5	1.4	6.6
	66	1.6	6.0	1.3	6.5
Minot	29	2.5	62.0	1.5	6.8
	31	2.8	16.0	1.7	10.0
Randolph	53	4.3	53.5	1.5	12.3
All Items		3.3	37.5	1.4	6.6

The above discussions, taken with Figures 1 and 5, clearly suggest that the demand for higher cost items will be easier to model. No reasonable probability model can be expected to represent the large turbulence of demand in Figure 5 and Tables 5 and 6 for low cost items. The pattern for low cost items suggests some kind of heuristic approach to demand prediction. We return to this idea shortly.

That the higher cost items are more important, in terms of supply investment, can be seen from Table 7 where we show statistics on demand value (DV) for the two cost groups. For each supply group and each cost category the number of items and the average demand value per item (in dollars) are shown. In every case the average demand value per item for the higher cost group is significantly larger than that for the lower cost group. Overall, the average demand value per item is approximately ten times as large for the former group as the latter. (The total demand value is $350,051.12 for the lower priced group and $2,157,227.23 for the higher priced group.)

All of the above data are shown to emphasize the fact that the higher cost items are the ones where we need a more accurate probability model for demand. A misrepresentation of the demand pattern for these

Table 7. Average Demand Value Per Item Versus Unit Cost

| Base | FSG | UC ≤ $5.00 | | UC > $5.00 | |
		Number of Items	DV/Item	Number of Items	DV/Item
Bentwaters	16	101	189.38	296	531.93
	59	1562	37.10	964	419.99
Dover	15	68	87.48	644	947.06
	59	1368	49.08	950	576.46
	66	104	25.18	269	841.06
Minot	29	157	113.22	191	479.23
	31	163	41.80	82	158.65
Randolph	53	2508	68.85	369	288.82
All Items		6031	$58.04	3765	$572.97

items can involve very large deficits or excesses in projected supply expenditures. Comparatively speaking, any possible errors of misrepresentation would be much less serious for the lower priced items. Given the turbulence associated with demand for the lower priced items and the extremely lower supply investment, a heuristic approach for demand prediction is unavoidable but certainly not too serious a compromise. A reasonable strategy to compensate for the heuristic model where predictions might vary considerably from actual demand would be to simply overstock the target lead time supply effectiveness. For example, set stock levels for a 0.88 or 0.90 probability rather than 0.84. Certainly, the supply investment would not significantly increase because of the low unit cost of these items. Further comments about this heuristic approach will be given in the next section.

Lead Time Experience Data

Next we turn to an analysis of empirical lead times. Figure 7 shows a typical lead time record for one item. Shown are the issue priority, the requisition date, receipt date, and order and shipping time (OST). Some apparent anomalies exist in the data. For example, is the second order a partial shipment of the first since they were ordered on the

Issue Priority	Requisition Date	Receipt Date	Order and Shipping Time
12	77-123	77-213	90
12	77-123	77-178	55
02	77-152	77-165	13
06	77-153	77-165	12
12	77-159	77-168	9
06	77-173	77-180	7
12	77-173	77-180	7
12	77-178	77-189	11
12	77-178	77-189	11
12	77-179	77-186	7

Figure 7. Typical Lead Time Record

same date but the second came in earlier? Are the two orders placed on day 178 of 1977 duplicate entries? (We assume in what follows that they are--consequently, one of the orders is deleted.)

To get a sufficiently large data set for a statistical analysis, we group all items in one FSG at one base and form a relative frequency diagram of the order and shipping times. Figure 8 shows the results grouped by routing identifier; the data set is the Bentwaters 16 account. Frequency diagrams are given for issue priorities 9-15 (referred to as priority group 3) and issue priorities 1-8 (priority groups 1 and 2). The diagrams for priority group 3 are very ragged and suggest a mixture of delivery modes. That is, in the Bentwaters-FGZ data for priority group 3 many orders arrive in 10-20 days, yet the mean OST is between 40 and 50 days. We suspect that occasionally priority group 3 items are mixed in and shipped with priority group 1 or 2 items. Both sets of OST data in Figure 8 for priority groups 1 and 2 agree fairly well with our expectation of a lead time distribution-- unimodal and skewed to the right.

It is often postulated that the gamma distribution is an applicable probability model for lead time distributions [7]. Our attempts to fit the gamma distribution to the data in Figure 8 and others have been unsuccessful. Indeed, we have been unable to fit the normal, lognormal or Weibull distribution to any empirical OST distribution considered in
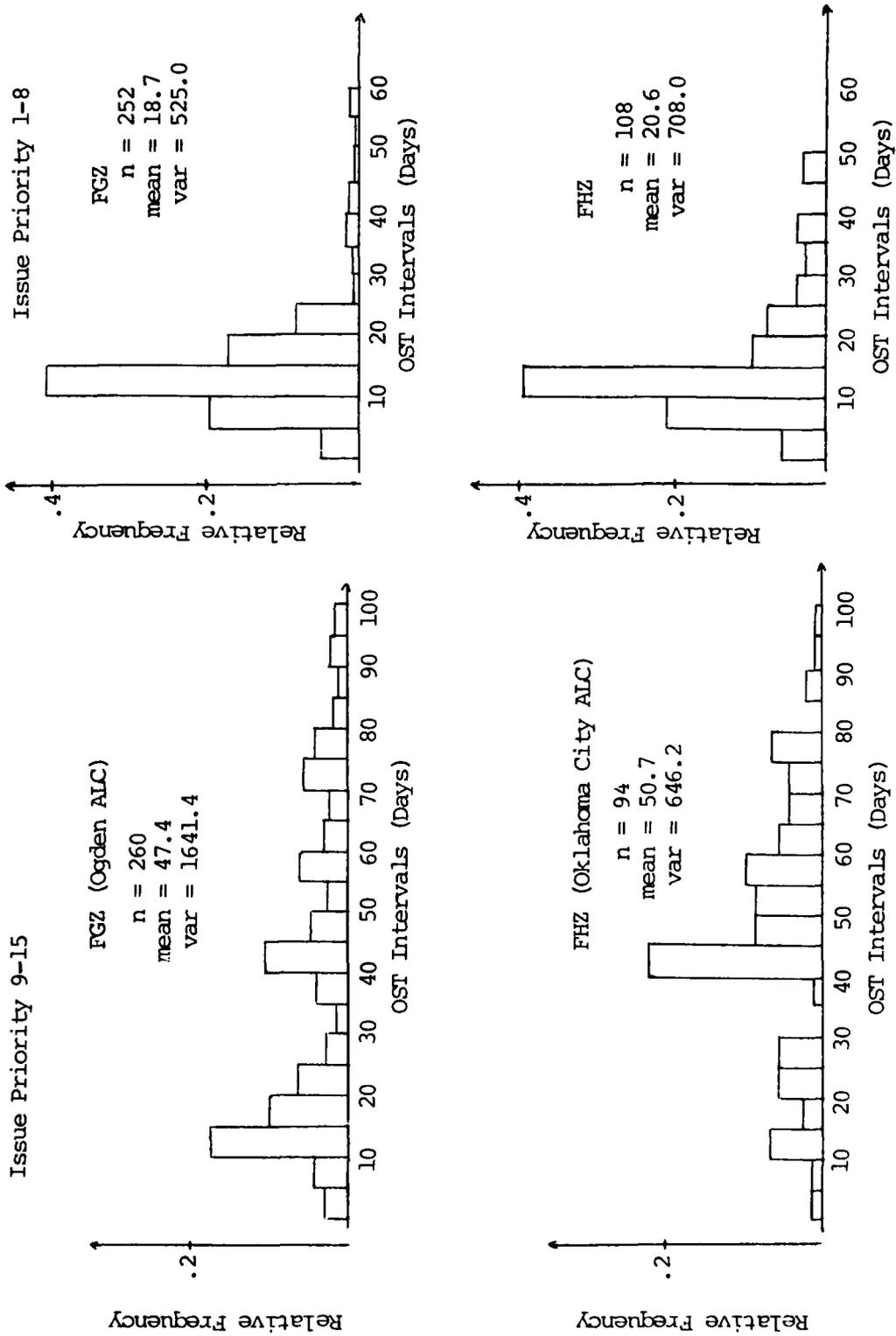
Figure 8. Relative Frequency Diagrams for OST by Routing Identifier and
Issue Priority Codes (Data Set: Bentwaters 16)

this study. Part of this problem may be due to the fact that we have grouped OSTs for a multitude of items from one base-resupplier combination as if they were from a common distribution. This may not be realistic. Not enough data are available on an individual item to study the OST distribution directly--hence, we have to resort to a consolidation. Operationally, a forecast of each individual item's OST seems impractical anyway.

As a result of the suspected anomalies cited in relation to Figures 7 and 8, a separate study has been initiated at the AFLMC to determine the mechanics of the lead time process. For our research, the procedure outlined in Section 2 for stochastic lead times will be used, that is, we determine the lead time demand by using the empirical lead time distribution. Since one of our primary objectives is to determine the sensitivity of the reorder point calculation to the variance of lead time, we think the available empirical lead time data, even with possible errors, is adequate for this purpose. The effect of variable lead time will be discussed in detail in Section 5.

The next section deals with fitting Models I and II to actual demand data.

SECTION 4
FIT OF PROBABILITY MODELS TO DATA

In this section we demonstrate how probability Models I and II of Section 2 for fixed lead time demand fit actual data. We also show how forecasts based on the models compare with actual demands in a fictitious lead time period. The data sets described in the last section provide an excellent opportunity to investigate the assumptions that are commonly used in inventory studies to model empirical data. Due to the paucity of data that existed in the past, these assumptions have seldom been subjected to real occurrences.

Several comments about probability modeling are in order before we show any results. First, before any modeling is attempted, the data should be examined for obvious anomalies and/or missing events--the last section addressed this issue. Second, it is not necessary that a probability model fit the data exactly to be useful; what we seek, at a very minimum, is a model that provides a reasonable fit to the data. In other words, the fitted model may help us understand a complex set of data without yielding a perfect fit to them. "Noise" in the data and/or the desire for a parsimonious model often lead us to accept a reasonable model rather than pursue a more accurate one. Third, a successful fit of a probability model does not necessarily imply that a unique set of circumstances produced the data. For example, the negative binomial probability model can be generated by 15 different stochastic processes [1]. Without other information, we could not say which process generated a set of data that may have been fitted with that distribution. Another example concerns the Poisson distribution. Events with low frequency in a large population can often be fitted by a Poisson distribution even when the probability of an event varies somewhat in the population (contrary to the classic Poisson postulates). Referring to the small number of events this empirical phenomena is sometimes called the law of small numbers [10]. The point to be made is that for purposes of predicting future events, such as lead time

demand, all that is necessary is that the model should fit the data—
we do not necessarily have to accept any particular set of assumptions
about the model to predict future occurrences reasonably well. With
these preliminaries we next attempt to fit the models described in
equations (1) and (2).

Our main objective is to model or characterize customer habits of
arrivals and units demanded, and then to use that characterization to
predict lead time demand. Referring to Figure 2 it is obvious that an
item's total demand for a specified period of time can be modeled as
some compound process. That is, a random number of customer arrivals
occur during the period $[0,\ell]$ and each customer demands a random number
of units. As in equation (1), if we assume the former has a Poisson
distribution and the latter a geometric distribution, then the total
demands in $[0,\ell]$ has the geometric-Poisson distribution. Next we show
how these two models (Poisson and geometric) fit actual arrival and
units demanded data. All goodness-of-fit tests will be performed using
a 0.95 confidence coefficient.

Fit of Poisson Distribution to Data

First we empirically study the distribution of the random
variable $N(\ell)$ in equation (1). If we have a forecasted lead time, of
say ℓ = seven days, we are interested in how the distribution of the
number of customers for a seven-day period, $N(7)$, agrees with the
Poisson assumption. To get empirical evidence for $N(7)$ we divide a
time period into nonoverlapping intervals of length seven days; we then
count the number of customer arrivals in each interval. To assess the
appropriateness of the Poisson distribution we next compare the obser-
vations to the expectations which result from the Poisson assumption.

For item 1 in Figure 1, which is from the Bentwaters account, we
form 37 seven-day intervals covering the observation period shown in
Figure 4. The observations (expectations) for $N(7) = 0$, 1, 2 are
30(29.8), 6(6.4) and 1(0.7), respectively. Using the Poisson dispersion
test as our goodness-of-fit test [9,14], we can conclude that the Poisson

23

expectations provide a very nice fit to the observations (at 0.95).
Item 1 in Figure 5 is also a Bentwaters item; its observations
(expectations) for N(7) = 0,1,2 are 24(26.0), 13(9.1) and 0(1.6).
Again, the fit is good. Table 8 summarizes the fit of the Poisson
distribution to all of the data we study. The column labeled N=0 gives
the number of items that had no customer arrivals for the respective
observation periods; the next column, N≥1, shows the number of items
that had one or more customer arrivals. The entry % Poisson, shows
the percentage of the N≥1 items that can be fitted reasonably well with
the Poisson distribution. (The N=0 items are excluded since an estimate
for λ is not obvious when there are no customer arrivals.) We conclude
that the Poisson distribution provides a reasonable fit to almost all
item's weekly arrival patterns. In addition, the quality of the fit
appears to be independent of unit cost. Because of the reproductive
property of the Poisson distribution [13] we expect it to provide a
reasonable fit for any value of ℓ, not just the seven day value used
here.

Table 8. Fit of Poisson Distribution

Base	FSG	UC ≤ $5.00			UC > $5.00		
		N=0	N≥1	% Poisson	N=0	N≥1	% Poisson
Bentwaters	16	12	84	89.3	30	265	92.5
	59	164	1368	90.6	111	852	93.2
Dover	15	22	46	100.0	191	453	94.7
	59	492	848	95.0	334	616	95.6
	66	29	74	95.9	95	174	92.0
Minot	29	35	117	93.2	56	135	90.4
	31	46	116	94.8	25	57	96.5
Randolph	53	304	1903	95.6	68	293	94.2
All Items		1104	4556	93.8	910	2845	93.9

Fit of Geometric Distribution to Data

The next random variable of interest is the units demanded per
customer, U_i in equation (1). Our analysis will necessarily be mostly

24

subjective due to the small number of customer demands that typically occur in an observation period. For example, item 1 in Figure 1 only has eight customer demands in the period shown. In Table 5 the average number of customer demands per item in a week is about 0.12 and so the average number of demands in an observation period of say, 37 weeks, would be approximately five. This small number of customer demands is insufficient for an objective goodness-of-fit test of the geometric assumption. For example, the observations (expectations) on item 1 in Figure 1 for $U = 1,2$ are 7(7.1) and 1(0.8), respectively. The fit certainly looks reasonable but we are unable to perform a χ^2 goodness-of-fit test because there are too few cells for comparison. (A minimum of three cells with expectations greater than one would be required to test the geometric assumption statistically.)

A possible way to get more data on units demanded per customer would be to increase the observation period to perhaps one or two years. We avoid this temptation because of reasons related to stationarity of the units per demand random process. It seems reasonable that longer periods would invite changes in such things as base missions, flying hour programs, maintenance and supply policies and monetary constraints—these changes could impact on a customer's demand habits. To minimize the effects of these possibilities we deliberately choose smaller observation periods. (This argument also applies to the previous material concerning the random variable $N(\ell)$.)

Because of the discussion in Section 3 on the characteristics of an item's units demanded per customer versus unit cost, we do not attempt to study the geometric distribution for the UC $\leq$ \$5.00 group. We will consider the low cost items in relation to the constant-Poisson distribution.

To study how the geometric distribution fits the random variable units demanded per customer, we consider just those items that had at least one customer arrival during the data collection period. Table 9 shows certain statistics on these items. There are 3765 items that cost

Table 9. Statistics on Units/Customer

Base	FSG	N≥1	% Items with Units/Customer = 1	% Items with Units/Customer = k>1
Bentwaters	16	265	47.5	15.5
	59	852	54.2	19.5
Dover	15	453	66.7	7.7
	59	616	59.6	11.7
	66	174	63.8	10.3
Minot	29	135	59.3	17.8
	31	57	38.9	33.3
Randolph	53	293	46.8	21.5
All Items		2845	56.5	15.4

over $5.00; 2845 had at least one customer demand during the respective observation periods. Of this latter group an average of 56.5% had a constant demand of one per customer (p = 0 in the geometric distribution) like item 2 in Figure 1. About 15% of the items had a constant demand greater than one, like the fourth item in Figure 1. This latter group can be modeled with the geometric distribution simply by rescaling to a demand of one per customer with a unit of issue of three, say. For all of these data, then, the geometric assumption is tenable on approximately 70% of the items with a demand. The remainder of the items have demand histories similar to items 1 and 3 in Figure 1. Next we examine demand data on them.

Since we do not have enough arrival data to study each item's units demanded per customer statistically, we resort to an aggregate analysis. Although not as desirable as an individual analysis, the aggregate approach can suggest a basic underlying model for customer demand.

If each item's units demanded per customer is a random drawing from a common geometric distribution, we would expect the probability distribution of units/customer to be of the form shown in Figure 9. This particular geometric distribution has a mean of two units per customer which is the approximate mean of our data (see Table 5 for UC>$5.00).
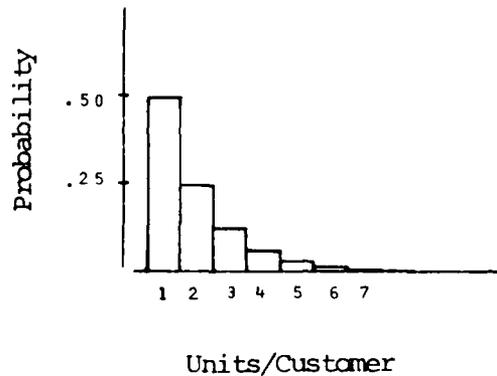
Figure 9. Typical Geometric Distribution

If units demanded per customer is geometrically distributed for each item (from a common distribution), then an aggregate plot of the data should resemble the distribution shown in Figure 9. That is, we look for a distribution with a mode at one unit that is skewed to the right. Figure 10 shows the aggregate data for each supply account studied. (The quantity n is the number of data points.) With the exception of the Bentwaters 16 account, each plot has the basic form of the geometric distribution--a maximum at one unit and generally decreasing to the right. Certainly, this analysis is not a proof of the geometric assumption, but it is highly supportive.

The above analyses of the random variables $N(\ell)$ and U_i in equation (1) suggest that the geometric-Poisson model is at least a reasonable approximation to the true underlying distribution of $D(\ell)$ in equation (1). We showed that the Poisson assumption was a very good representation for customer arrivals. Although the geometric assumption for the U_i's was not substantiated in a statistical sense, it was shown to have several appealing properties that are consistent with the data: (a) it does allow for a constant U_i (with $p = 0$) which is a characteristic of a large percentage of the items; (b) it can represent a variable demand (with $p > 0$); and (c) its probability distribution has the same general shape as actual demand data. The "acid test" of the geometric-Poisson model, though, is how well it works in an operational sense; that is, how well it predicts future demands.

27
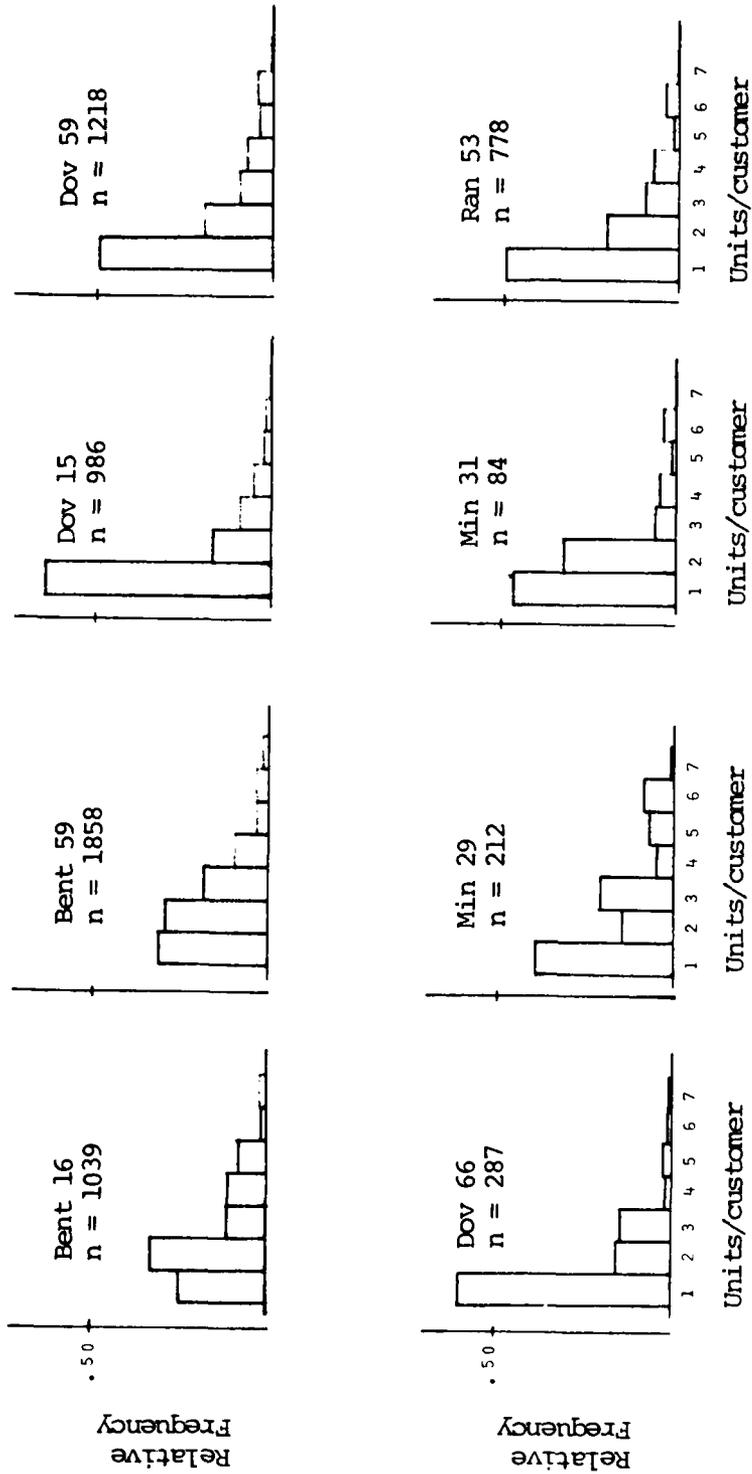
Figure 10. Relative Frequency Diagrams for Units Demanded per Customer

28

Operational Test of Geometric-Poisson Model

To study the effectiveness of the geometric-Poisson model in fore-
casting future demands, two adjacent time periods are considered. The
first period is a data collection period and the second is a holdout
period. We let the holdout period correspond to a fictitious lead time
period. In essence, model parameters (λ and p for the geometric-Poisson
distribution in Table 1) are estimated from data in the observation or
data collection period, and then "lead time demand" is predicted for
the holdout period. If the predicted demand agrees reasonably well
with the observed demand, then the geometric-Poisson model can be used
to model lead time demand and subsequently set reorder points.

Computing the reorder points is one of our main goals. Since a
reorder point is supposed to satisfy a certain percentage of lead time
demand, say 84%, the cumulative form of the geometric-Poisson distribu-
tion can be used to determine the appropriate quantity. This process
was explained in Section 2.

The observation and holdout periods for each supply account were
shown in Figure 4. The accounts at Bentwaters have holdout periods of
seven weeks; the other accounts have holdout periods of three weeks.
These values correspond roughly to the average lead time from the bases
to their respective principal resuppliers. The observation periods for
the Bentwaters and other supply accounts are 37 weeks and 26 weeks,
respectively. Thus, for Bentwaters we use 37 weeks of observed data to
predict usage in the next 7 weeks; for the CONUS bases we use 26 weeks
of observed data to predict 3 weeks of usage. We would have preferred
a longer observation period for the Bentwaters data (a ratio of 26 to 3
in comparing the observation to the holdout period) but this was not
possible due to missing data and the other reasons cited before.

The next task is to actually predict demand in the holdout or
fictitious lead time period. Since we are primarily interested in the
lead time distribution to set reorder points we make predictions that
should cover 84%, say, of the second period demands. We can compare

29

these predictions to observations and then judge the merits of the
geometric-Poisson distribution for this specific purpose.

Before showing summary results, some specific cases are given.
Table 10 shows a few examples of how the predictions and observations
compare. We arbitrarily choose five items from the Dover 15 account.

Table 10. Predictions and Observations for the Geometric-Poisson

Federal Stock Number	Observed Demands Period 1	Predicted Demands Period 2 (@ .84)	Observed Demands Period 2
1560000094144	7	2	1
1560001030911	23	4	4
1560002254290	3	0	0
1560004773610	4	1	2
1560004981175	10	3	1

The predictions are made by taking the mean and variance of an item's
weekly demands throughout the observation period, and then estimating
the parameters of the geometric-Poisson. The cumulative distribution
is then formed and a prediction that will satisfy 84% of the forecasted
demand is made. The value of ℓ in the geometric-Poisson is set to
three for this example. Essentially we are saying that if the reorder
point is set to this value, 84% of the demands will be satisfied, on
the average. For example, for the first item in the table, if we
reorder when the inventory position reaches two units, we should
satisfy all demands for 84% of the lead time cycles or we say the
probability if 0.84 that we will satisfy all demands for a single lead
time cycle. It is important to note that the 0.84 applies just to a
lead time cycle and not to the effectiveness of the supply system for
all time.

For the first item, the actual number of demands in the holdout
period totaled only one. Our reorder point would have been sufficient,
then, if the lead time had been three weeks. Similarly, item 2 would
have had sufficient stock to satisfy all demands during the second

period. The fourth item's reorder point would have been too low to satisfy lead time demand. If the geometric-Poisson is a reasonable model, we would expect about 84% of the reorder points among a large group of independent items to be sufficient for the actual demands in period 2.

Table 11 summarizes how the predictions compare with observations for the individual accounts. Overseas bases like Bentwaters are stocked for 95% of forecasted lead time demand; the CONUS bases are stocked for 84%. These percentages are specified by current Air Force policy. For each account we show the number of items, percent of items for which the respective reorder points would have been sufficient, and the resulting dollar investment in lead time spares. For example, we made predictions on 247 items in the Bentwaters 16 account; of these,

Table 11. Summary Statistics for the Geometric-Poisson (G-P) and Current System Reorder Points (UC $5.00)

Base	ERRC	Number of Items	Percent Sufficient		$ Investment	
			G-P	Current	G-P	Current
Bentwaters	16	247	91.1	95.5	44,440.25	57,341.77
	59	811	93.2	96.4	106,869.38	152,949.70
Summary		1058	92.7	96.2	151,309.63	210,291.47
Dover	15	444	83.1	88.3	38,944.85	66,251.58
	59	587	86.2	90.5	32,776.22	50,255.56
	66	167	89.2	92.2	15,629.09	25,661.03
Minot	29	129	85.3	88.4	5,340.59	8,664.07
	31	50	94.0	94.0	6,25.11	1,245.25
Randolph	53	258	87.6	88.8	5,042.00	7,232.30
Summary		1635	86.1	89.7	98,367.92	159,314.29

reorder points corresponding to the predictions would have been sufficient on 91.1% of the items. That is, demand during the holdout period on 91.1% of the items was less than or equal to the 0.95 prediction. The dollar investment for these hypothetical lead time spares would have

been $44,440.25. The 247 items are those items with at least one customer arrival during the observation or data collection period. This number is less than the number shown in Table 8 since we removed those items with a variance-to-mean ratio, of total units demanded, greater than seven. For reasons presented before, we believe some items contain outliers and we select a cutoff on the VMR of seven to exclude those items. A VMR of seven is the approximate 95th percentile value for all items with a unit cost over $5.00. Each data set has a number of items removed for this reason.

For comparison, the performance of the current Air Force system is also shown in Table 11. In the current system, an item's reorder point is its mean lead time demand plus a safety stock. The mean lead time demand is nothing more than the daily demand rate times the forecasted lead time period (the holdout period here). The safety stock for CONUS bases is the maximum of 15 days of demand or one standard deviation of lead time demand; for overseas bases, the safety stock is the maximum of 30 days of demand or two standard deviations of lead time demand. In both cases, the reorder point is always truncated to the next lower integer value. For example, a calculated reorder point of 3.1 would be truncated to 3.0, and so would 3.9. As mentioned before, in computing the standard deviation of lead time demand, a VMR of three is used. For example, if the mean lead time demand is expected to be five units, then the standard deviation of lead time demand would be the square root of 15. In Table 11 we see that the current system for the Bentwaters 16 account satisfies 95.5% of the holdout period demands and costs $57,341.77. In this case the current system is a better predictor since its forecast is closer to the target lead time supply effectiveness of 95.0%.

From the summary line in Table 11 for the two Bentwaters accounts we see the respective percentages and costs. The observed reorder point effectiveness for the current system is closer to the target of 95% than is the effectiveness via the geometric-Poisson model. The

32

investment in lead time spares in considerably higher, though, for the current system. The summary line for the last three base accounts is more favorable to the geometric-Poisson model, both in terms of being closer to the target of 84% and having a smaller investment.

As an overall summary of these results we observe the following: (a) the geometric-Poisson missed the target by -2.3% for the Bentwaters accounts and was over by 2.1% for the CONUS accounts--the average error here is -0.1%; (b) for the current system the average error can be seen to be +3.4%; (c) the lead time spares investment for the geometric-Poisson model compared to the current system is 72.0% for Bentwaters and 61.7% for the other accounts. Based on these two measures, we think the geometric-Poisson model is a more reasonable and certainly more cost effective model than the current system. We showed before that the current system's assumptions of normally distributed lead time demand and a constant VMR of three for every item are highly questionable. The assumptions are very easy to implement but evidently they are not very representative of the actual processes which generate lead time demands.

It is important to note that these results are based on a particular set of bases and observation and holdout periods. We would not expect these percentages and dollar investments to be constant for every base and every set of observation and holdout periods. Indeed, the percentages and dollar investments are random variables. However, due to the large samples and variety of supply accounts studied here, we would expect these general results to be duplicated for other base accounts and data periods.

A Consideration for Implementation

Due to the large size of a typical Air Force base supply account, any candidate probability model should be one that is relatively easy to implement. On the surface, the form of the geometric-Poisson distribution shown in Table 1 looks very complicated and suggests large amounts of machine time to compute the individual probabilities. To the

contrary, for typical values of the mean and VMR of lead time demand shown here, the computer time is not too excessive. However, in order to reduce the computer time to an absolute minimum, a look-up table can be constructed to give the reorder point directly. Table 12 shows such a table. Each entry in the table is formed from the cumulative geometric-Poisson distribution for a particular mean and VMR of lead time demand. For example, an item with a mean and VMR of 1.3 and 2.2, respectively, would have a reorder point (at .84 probability) of 2.0 units. This particular table covers about 90% of the items, in practice a bigger table would be required.

Table 12.   Reorder Points for Geometric-Poisson Probability
            Model (at 84% Confidence Level)

· Constant lead time

Variance/Mean of lead time demand	.1	.5	.9	1.3	1.7	2.1	2.5	2.9								
5.0	0	0	0	1	1	2	3	3	3	4	4	5	5	5	6	6
	0	0	0	1	2	2	3	3	3	4	4	5	5	5	6	6
	0	0	0	1	2	2	3	3	3	4	4	5	5	5	6	6
	0	0	0	1	2	2	3	3	3	4	4	5	5	5	6	6
	0	0	0	1	2	2	3	3	3	4	4	5	5	5	6	6
4.0	0	0	0	1	2	2	3	3	3	4	4	4	5	5	5	6
	0	0	0	1	2	2	3	3	3	4	4	4	5	5	5	6
	0	0	0	1	2	2	3	3	3	4	4	4	5	5	5	6
	0	0	1	1	2	2	3	3	3	4	4	4	5	5	5	6
	0	0	1	1	2	2	3	3	3	4	4	4	5	5	5	6
3.0	0	0	1	1	2	2	2	3	3	4	4	4	5	5	5	5
	0	0	1	1	2	2	2	3	3	4	4	4	4	5	5	5
	0	0	1	1	2	2	2	3	3	3	4	4	4	5	5	5
	0	0	1	1	2	2	2	3	3	3	4	4	4	5	5	5
	0	0	1	1	2	2	2	3	3	3	4	4	4	5	5	5
2.0	0	0	1	1	2	2	2	3	3	3	4	4	4	4	5	5
	0	0	1	1	2	2	2	3	3	3	3	4	4	4	5	5
	0	0	1	1	2	2	2	3	3	3	3	4	4	4	5	5
	0	0	1	1	2	2	2	2	3	3	3	4	4	4	4	5
	0	0	1	1	1	2	2	2	3	3	3	3	4	4	4	5
1.0	0	0	1	1	1	2	2	2	3	3	3	3	4	4	4	4

Mean lead time demand

It is interesting to note how stable the reorder point is to
changes in the mean and VMR of lead time demand. For instance, for a
mean lead time demand of 1.7, a reorder point of 3.0 applies for any
VMR between 1.0 and 5.0. This same reorder point is seen to apply to
other means and VMRs also.

Another interesting item to note in the table is how the reorder
point, for a particular low value of the mean, say 0.5, decreases for
large values of the VMR. Intuition would suggest that the reorder point
should always increase for a large VMR, like for a mean of 2.9 in the
table. We refer to the cause of this action as "origin blocking."
Figure 11 shows graphically the phenomenon. As the VMR increases, for
a particular value of the mean, the variance of the distribution also
increases and we would expect the distribution to flatten out on either
side of the mean. As can be seen in the figure, the distribution does
spread out to the right of the mean, but on the left side it is
restricted from taking on negative values and so the probabilities
actually increase or bunch up near the origin. This blocking action
affects the reorder point computation in the following way.

Suppose we desire a reorder point, at the 0.75 level, on an item
with a mean lead time demand of 0.5. From Figure 11 we see that if
the item's VMR is 1.0, then its reorder point would be one unit. If
its VMR is 4.0, however, then its reorder point would be zero. We
suspect that this so-called blocking action is also characteristic of
other discrete distributions, such as the negative binomial.

Operational Test of Constant-Poisson Model

The analyses presented so far have been directed at modeling the
customer arrival and demand processes in an exact sense. Given the
nature of the units per demand data as illustrated in Figure 1 and
Table 9, that is, constant demand on many items and a small variance
on others, it is natural to consider an approximation to the demand
process. In general, we are interested in how robust the reorder point
calculation is to differences in the form of the units per demand

35
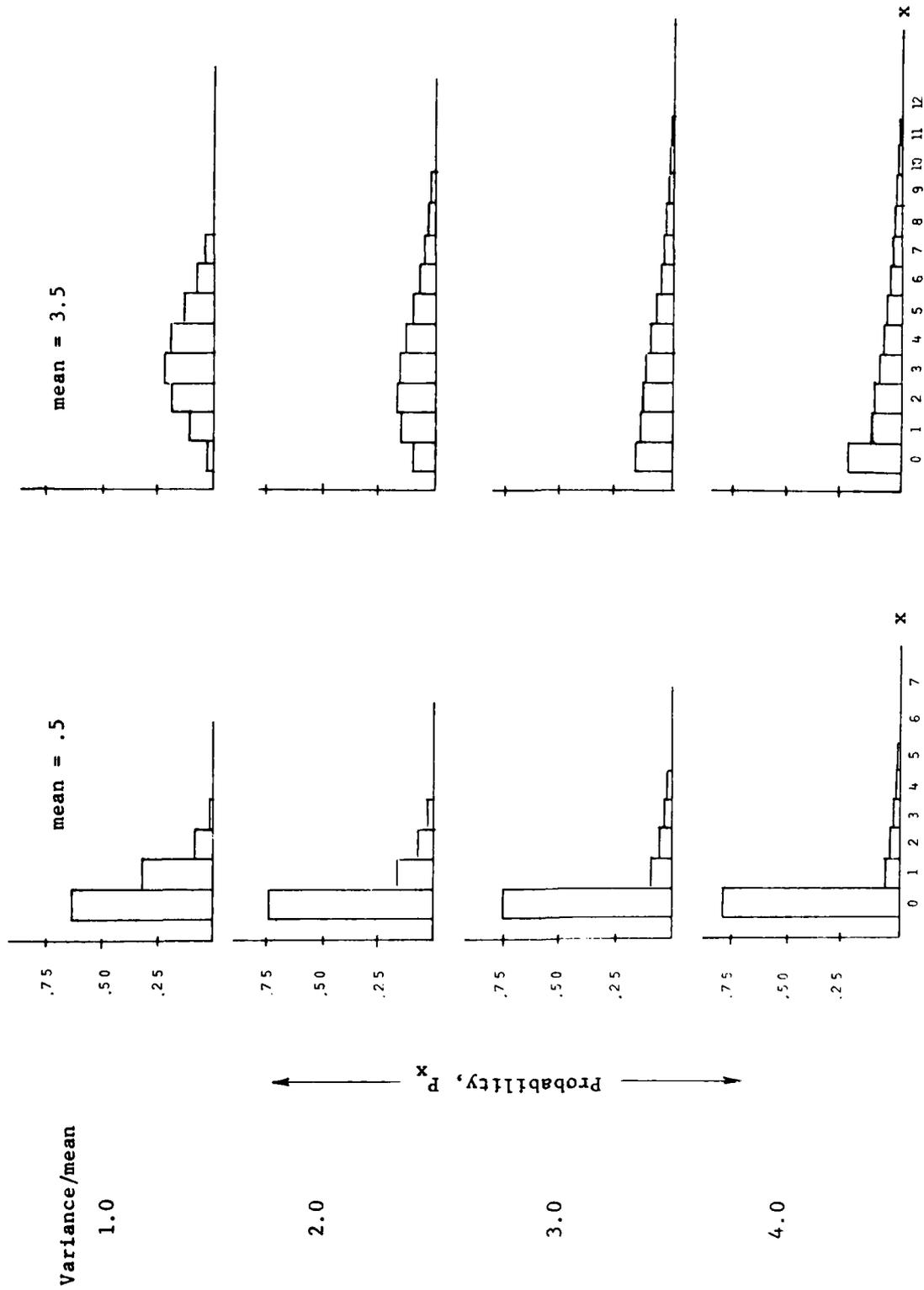
Figure 11. Geometric-Poisson Probabilities, P_x

36

process. Our specific interest is in the question—what is the lead time supply effectiveness and spares investment for an approximate model such as the constant-Poisson (Model II) as compared to the geometric-Poisson model? Here the constant quantity is taken to correspond to the mean units demanded per customer. Clearly this model is an approximation to the demand process, but it may be adequate for prediction purposes. This is one approximation to a compound-Poisson process, like the geometric-Poisson, that is sometimes used in practice [11]. (Another approximation that is quoted as being useful is the simple Poisson process [11]. For our data where many items have a demand other than one, the constant-Poisson model is a better candidate.)

Table 13 gives a look-up table that can be used to determine an item's reorder point based upon this model (reorder points are computed from the cumulative distribution function for the constant-Poisson model).

Table 13. Constant-Poisson Model Reorder Points
(at 84% Confidence Level)

· Constant lead time

```
          4.0 ┬0   4   4   8   8   8  12  12  12  15  15  15  20  20  20  24
               0   4   4   8   8   8  11  11  11  14  14  14  19  19  19  23
          3.5 ┬0   4   4   7   7   7  11  11  11  13  13  13  18  18  18  21
               0   3   3   7   7   7  10  10  10  12  12  12  16  16  16  20
          3.0 ┬0   3   3   6   6   6   9   9   9  11  11  11  15  15  15  18
               0   3   3   6   6   6   8   8   8   9   9   9  14  14  14  17
          2.5 ┬0   3   3   5   5   5   8   8   8   9   9   9  13  13  13  15
               0   2   2   5   5   5   7   7   7   8   8   8  11  11  11  14
          2.0 ┬0   2   2   4   4   4   6   6   6   7   7   7  10  10  10  12
               0   2   2   4   4   4   5   5   5   6   6   6   9   9   9  11
          1.5 ┬0   2   2   3   3   3   5   5   5   6   6   6   8   8   8   9
               0   1   1   3   3   3   4   4   4   5   5   5   6   6   6   8
          1.0 ┬0   1   1   2   2   2   3   3   3   4   4   4   5   5   5   6
               └────┼───────┼───────┼───────┼───────┼───────┼───────┼────
                   .5      1.0     1.5     2.0     2.5     3.0     3.5     4.0
```

Average number of units per demand

Expected number of customer arrivals during lead time

The two parameters necessary to enter the table are the expected number of customer arrivals during lead time and the expected units demanded per customer. For example, an item with expected values of 2.0 and 1.5

37

for the number of customer arrivals and units demanded per customer, respectively, would have a reorder point of 5.0 units (at an 84% confidence level). Parameter values are estimated using historical information from a data collection period.

For each item in an account, we can use the look-up table to set a reorder point for a fictitious lead time period of, say, 21 days. We can then compare the observed demands to the hypothetical reorder point and determine how effective it would have been in satisfying lead time demand. This operational test is exactly as was described for the geometric-Poisson model. Table 14 shows the lead time supply effectiveness and spares investment for each account. (As before, the Bentwaters accounts are stocked at 95% and the CONUS accounts at 84%.)

Table 14. Summary Statistics for the Constant-Poisson Reorder Points (UC > $5.00)

Base	FSG	% Sufficient	$ Investment
Bentwaters	16	93.5	42,671.48
	59	94.8	117,799.87
Summary		94.5	160,471.35
Dover	15	81.3	37,161.53
	59	85.2	28,690.77
	66	88.0	14,684.13
Minot	29	82.2	4,466.03
	31	94.0	475.24
Randolph	53	86.4	4,107.07
Summary		84.6	89,584.77

The number of items and the observation and holdout periods are the same as were used in Table 11. An inspection of Tables 11 and 14 shows how the constant-Poisson model compares to the geometric-Poisson and current system models. It is easy to see that the constant-Poisson compares very favorably to the geometric-Poisson model. The former is actually closer to the lead time supply effectiveness than the latter;

the spares investments for both models are about the same. Comparing the constant-Poisson to the current system we see a dramatic improvement, both in terms of being closer to the target effectiveness and in terms of much lower spares investments (this same improvement was noted for the geometric-Poisson model). That the constant-Poisson model fares so well here is not surprising given that about 70% of the items studied have a constant demand (Table 9). One of the reasons for considering this constant approximation to the units demanded per customer is for possible application to the items with a unit cost of $5.00 or less. We next investigate those items.

Even though the more expensive items (UC · $5.00) represent larger inventory investments for lead time spares, the low cost items can create problems for the supply manager as well. It is important, therefore, to analyze their demand patterns in some detail with the objective of finding a model that is consistent with the model for the larger cost items.

As illustrated in Figure 5 for some typical items with a UC ≤ $5.00, empirical data on units demanded per customer suggest that our chances of fitting a standard distribution, such as the geometric, to them is about nil. The items in Figure 5 do appear to have some constancy, though, and it is this observation that leads us to consider the constant approximation for units demanded per customer. Item one in the figure has a majority of customer demands of 63, item two a majority of 4, item three's demands are all for 3, and item 4 is a mixture of mostly 5s and 15s. For all UC ≤ $5.00 items and the same observation periods described before, 56.4% of 4933 items with at least one customer arrival have a strictly constant demand, like item three in the figure. Coupled with the results in Table 8 showing the applicability of the Poisson distribution in describing the number of customer arrivals per unit time, we take the constant-Poisson model as an approximation to the total demand process.

In Table 15 we show how the constant-Poisson and current system compare for items with a UC ≤ $5.00 (reorder points for the constant-

39

Table 15. Summary Statistics for the Constant-Poisson (C-P)
and Current System Reorder Points (UC $\leq$ $5.00)

Base	FSG	Number of Items	Percent Sufficient		$ Investment	
			C-P	Current	C-P	Current
Bentwaters	16	89	92.1	89.9	5,034.23	4,994.79
	59	1401	93.4	93.4	15,951.70	16,695.32
Summary		1490	93.3	93.2	20,985.93	21,690.11
Dover	15	46	63.0	71.7	515.37	602.92
	59	875	80.7	84.5	4,442.32	6,107.03
	66	75	84.0	86.7	188.58	305.60
Minot	29	122	74.6	80.3	1,139.97	1,606.43
	31	117	88.9	88.9	348.46	594.24
Randolph	53	2210	79.4	82.1	18,940.19	22,910.12
Summary		3445	79.8	82.8	25,574.87	32,126.34

Poisson are from tables like 13). For the Bentwaters accounts both models look equivalent. The constant-Poisson has a lower cost for the CONUS accounts but its percent sufficient statistic is a little less than the current system's. Given that the percent sufficient statistic is farther from the target for the CONUS bases we cannot say that the constant-Poisson model is an improvement over the present system; on the other hand, given the results at Bentwaters and for the CONUS bases, the constant-Poisson seems to be almost as good as the current system.

The approximation is not expected to be as good for these cheaper items as was evident for the higher cost items. A comparison of Figures 1 and 5 shows that the constant assumption for units demanded per customer is more likely to be a better approximation for the higher cost items than the cheaper group. As discussed before, we view the constant-Poisson mainly as a heuristic development for these low cost items. Even if a target effectiveness of .90 was required to obtain .84 in practice, the spares investment would probably not be that much more expensive than the investment shown in Table 15.

Summary

To this point we have shown how the geometric-Poisson and constant-Poisson models fit actual data for items that cost over $5.00. For fictitious lead time periods, both models give about the same results and both represent significant improvements over the current system. The geometric-Poisson is a more exact model than the constant-Poisson model which is clearly an approximation. Although the former model is more aesthetically pleasing, the latter model provides comparable predictions.

Based on a comparison of the constant-Poisson and the current system for items with a UC $\leq$ $5.00, it seems reasonable to consider the constant-Poisson model for all items, regardless of unit cost. It would provide savings in spares investment for the higher cost items and would not be any more costly than the present system for the cheaper items.

Indeed, for the data used here, we can use Tables 11, 14 and 15 to show that the total dollar investment, regardless of unit cost, is $423,422.71 for the current system and $296,612.92 for the constant-Poisson model. The constant-Poisson model's cost is 70.1% of the current system's. The lead time reorder point supply sufficiencies are 94.4% (current), 93.8% (constant-Poisson) for Bentwaters; for the CONUS accounts the percentages are 85.0 (current) and 81.3 (constant-Poisson). We doubt that the slightly higher percentages for the current system would account for the higher investments for that system. Although not performed in this study, an alternate procedure would examine the differences in lead time supply performance for the same dollar investments in reorder point spares. Based on the results here, we would expect the constant-Poisson model to provide the higher supply performance.

Next we examine the effect of variable lead time on both models.

SECTION 5

THE EFFECT OF VARIABLE LEAD TIME

In this section we allow lead time to be a random variable in order
to determine what impact its variability has on the reorder point
computation. We view this effort as mostly exploratory in nature at
this point. The current study being conducted at the AFLMC on the
mechanics of the lead time process will undoubtedly shed new light on
the lead time data we study and possibly will answer some of the
questions posed earlier about them. Although our analysis is cursory,
we do demonstrate a methodology to study the effect of variable lead
time. To illustrate the technique, we examine priority group three
(routine) shipments only and consider just those items with a unit
cost over $5.00. In keeping with our concern to study the exactness
of all random processes, we concentrate on the geometric-Poisson model.

The analysis we show will be numerical in the sense described
relative to Model III in Section 2. That is, we use the empirical
distribution of lead time, $g(\ell)$, to determine the marginal distribution
of lead time demand, $f(d)$, in equation (4). From $f(d)$ the cumulative
distribution of lead time demand is formed and then a reorder point is
determined. Before showing how this process compares with the results
for a fixed lead time model, we first give some descriptive statistics
about the lead time process and about demands associated with a variable
lead time.

Statistics on Lead Time

Figure 8 showed relative frequency diagrams of empirical lead times
for several base-resupplier combinations. In Table 16 we show the mean
and variance-to-mean ratio of lead time for each base and its major
resupplier (the routing identifier codes are FGZ--Ogden ALC; S9E--Defense
Electronic Supply Agency; FPZ--San Antonio ALC; and S9I--Defense
Industrial Supply Center). All resuppliers were taken for the Minot
accounts since no one resupplier seemed dominant. The unit of time is

Table 16. Statistics on Lead Time

Base	FSG	n	Resupplier	Mean	VMR
Bentwaters	16	263	FGZ	6.4	2.6
	59	1051	S9E	6.6	1.4
Dover	15	2206	FPZ	2.9	0.5
	59	1597	S9E	2.8	0.5
	66	211	FPZ	3.1	0.5
Minot	29	806	All	2.7	0.6
	31	378	All	2.8	0.5
Randolph	53	457	S9I	3.4	0.7

one week. The quantity n is the number of lead time cycles for the
respective account-resupplier combinations. In computing the
statistics, all observations for Bentwaters greater than 17.1 (120 days)
are deleted; for CONUS accounts the corresponding value is 7.7 (54 days).
This coincides with a standard policy currently in use by the Air Force
to compute statistics on the lead time process. The mean and VMR are
much larger for the Bentwaters accounts than the CONUS ones.

Another variable of interest in this section is the VMR of total
demands associated with a variable lead time. This is the quantity
that the Air Force is currently assuming to be three for each item in
the safety stock computation. If there were ample lead time cycles
on each item, we could study the VMR of lead time demand directly.
Instead, we estimate the VMR by using equations (5) and (6) of
Section 2. Dividing equation (6) by (5) we can form the ratio and
obtain:

$$VMR_{(var)} = VMR_{(fix)} + VMR_{(L)} E(U) E(N). \tag{7}$$

In (7), $VMR_{(var)}$ is the VMR of demand for a variable lead time, $VMR_{(fix)}$
is the VMR of demand associated with a fixed unit of time, say one week,
$VMR_{(L)}$ is the VMR of the lead time process, $E(U)$ and $E(N)$ are as
defined before. Algebraically, $VMR_{(fix)}$ corresponds to the first term
in equation (6) (since $V(L)$ equals zero in this case) divided by
equation (5). To compute (7) on each item, we estimate $VMR_{(fix)}$, $E(U)$

43

and $E(N)$ from sample data for an observation period, say of 26 weeks (we use the periods in Figure 4). In addition, the VMR of lead time, $VMR_{(L)}$, is taken from Table 16. A plot of the different variance-to-mean ratios, $VMR_{(var)}$, for any one base account would look similar to Figure 6 where we showed a relative frequency chart of $VMR_{(fix)}$ for the Dover 59 account. It would be evident that there is considerable variability in the VMRs of demand associated with variable lead time, as was indicated for $VMR_{(fix)}$. Table 17 gives the median and 95th percentile of $VMR_{(var)}$ for the individual accounts. Comparing these entries to those in Table 6, we see that the medians are approximately the same but the 95th percentiles in Table 17 suggest that the empirical distributions here are much more skewed to the right.

Table 17.   Variance-to-Mean Ratio of Total Demands
During a Variable Lead Time

Base	FSG	Median	95th Percentile
Bentwaters	16	2.0	10.0
	59	1.5	9.0
Dover	15	1.3	5.5
	59	1.4	8.5
	66	1.4	8.0
Minot	29	1.6	9.5
	31	2.1	10.0
Randolph	53	1.7	15.0
All Items		1.5	9.0

Without having access to the analysis that established the current policy of a constant VMR of 3.0 for each item, we have to question the validity of that policy, given the results here. It seems inappropriate to use a constant of 3.0 for every item when 50% of the items have an estimated VMR less than 1.5. Indeed, 78.0% of the items shown here have a VMR less than 3.0! We hasten to add that the estimates referred to in Table 17 are for items that cost over $5.00; although the VMR of

the lower cost items would most likely be larger, we think it is more appropriate to study the higher cost items because of the larger implied inventory investment. With these comments about the variability of lead time and the associated demands, we now investigate the effect on the reorder point.

Impact of Variable Lead Time on Reorder Point

The real concern in this section is whether or not the reorder point computation is sensitive to the variability of lead time. To address this issue we arbitrarily take items from the base account-resupplier combination, Dover 59-39E, and see how the reorder point computation differs from our work in the last section. Specifically, we use $g(\ell)$ for the Dover 59-39E items (Figure 12) to determine $f(d)$ of equation (4); we then use $f(d)$ to determine the appropriate reorder points. Since our main concern is to see how the variability of lead time effects the reorder point, we have not omitted lead time data beyond 54 days (the Air Force convention) in Figure 12. Instead we have taken all data to the 95th percentile. This produces a larger variability and should represent somewhat of a worse case, at least in the sense of testing its effect on the reorder point. For later use we show in Figure 12 the fit of the gamma distribution to the empirical data. In a statistical sense the fit is inadequate (using the χ^2 goodness-of-fit test at $\alpha = 0.05$).

We first consider the geometric-Poisson model and allow for the observed variable lead times illustrated in Figure 12. That is, for the process outlined in Model III we take $f(d|\ell)$ as the geometric-Poisson and use $g(\ell)$ from Figure 12. For different combinations of the mean and VMR of lead time demand we compute $f(d)$ as in equation (4). We then form the cumulative distribution function to get the reorder point to satisfy 84% of the lead time demand. Table 18 shows the resulting reorder points. This table can be compared to the results for the fixed lead time model in Table 12. Clearly there are no dramatic changes—a few differences of one unit exist for some of the

-46-

n = 1639
mean = 20.9
var = 120.3

observed
expected (gamma distribution)

Frequency of Occurrence
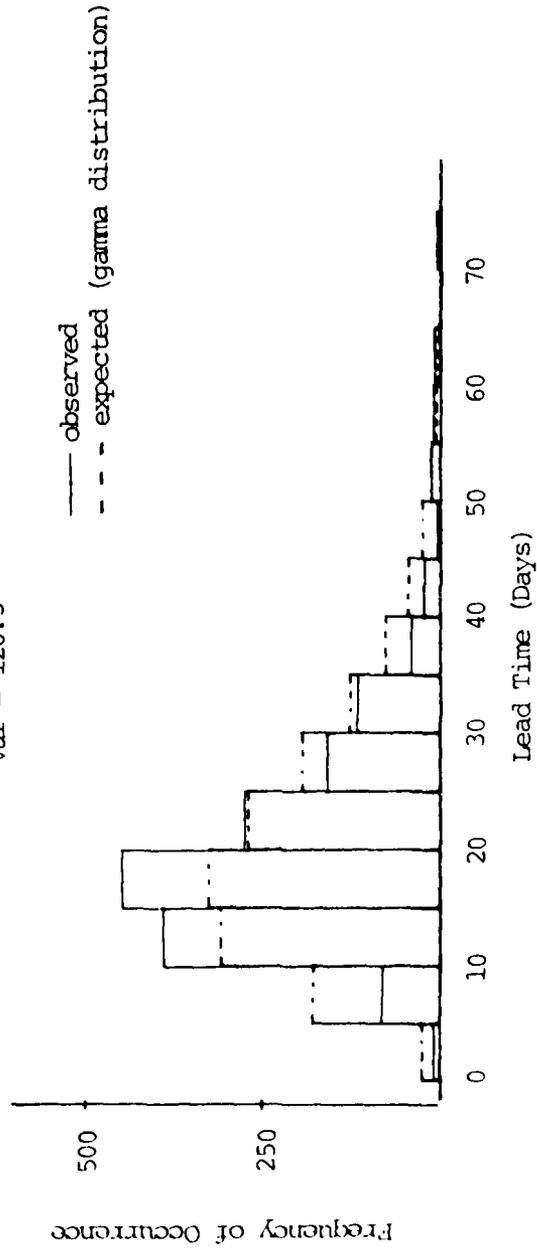
Lead Time (Days)

Figure 12. Empirical Lead Time Data for Dover 59-S9E

Table 18. Reorder Points for Geometric-Poisson Probability
           Model (at 84% Confidence Level)

· Variable lead time
· Dover 59-S9E data

```
Variance/mean of
lead time demand

5.0 ┤0  0  0  1  1  2  2  3  3  4  4  5  5  5  6  6
     0  0  0  1  1  2  3  3  3  4  4  5  5  5  6  6
     0  0  0  1  1  2  3  3  3  4  4  5  5  5  6  6
     0  0  0  1  1  2  3  3  3  4  4  5  5  5  6  6
     0  0  0  1  2  2  3  3  3  4  4  5  5  5  6  6
4.0 ┤0  0  0  1  2  2  3  3  3  4  4  5  5  5  6  6
     0  0  0  1  2  2  3  3  3  4  4  4  5  5  6  6
     0  0  0  1  2  2  3  3  3  4  4  4  5  5  5  6
     0  0  0  1  2  2  3  3  3  4  4  4  5  5  5  6
     0  0  1  1  2  2  2  3  3  4  4  4  5  5  5  6
3.0 ┤0  0  1  1  2  2  2  3  3  4  4  4  5  5  5  6
     0  0  1  1  2  2  2  3  3  4  4  4  5  5  5  6
     0  0  1  1  2  2  2  3  3  3  4  4  5  5  5  5
     0  0  1  1  2  2  2  3  3  3  4  4  4  5  5  5
     0  0  1  1  2  2  2  3  3  3  4  4  4  5  5  5
2.0 ┤0  0  1  1  2  2  2  3  3  3  4  4  4  5  5  5
     0  0  1  1  2  2  2  3  3  3  4  4  4  5  5  5
     0  0  1  1  2  2  2  3  3  3  4  4  4  4  5  5
     0  0  1  1  2  2  2  2  3  3  3  4  4  4  5  5
     0  0  1  1  1  2  2  2  3  3  3  4  4  4  5  5
1.0 ┤0  0  1  1  1  2  2  2  3  3  3  4  4  4  4  5
     └──┼─────┼─────┼─────┼─────┼─────┼─────┼─────┼──
       .1    .5    .9   1.3   1.7   2.1   2.5   2.9

                Mean lead time demand
```

mean-VMR combinations; however, these are primarily at the fringes of
the regions of constant reorder points. For example, for a mean of
2.1 and for VMRs of 1.0 through 1.4, the reorder point in Table 18 is
3.0. In Table 12 for the same mean, the reorder point is 3.0 for VMRs
of 1.0 through 1.8.

Another way to compare the fixed and variable lead time models is
to use both for predictions in fictitious variable lead time cycles.
Using the same observation periods for both we estimate model parameters
for the two models and then make predictions (that is, we set hypothetical
reorder points) for simulated lead time periods. For Model I we take
reorder points from Table 12 where a fixed lead time is assumed. For

47

Model III we take reorder points from Table 18 where a variable lead time is assumed. The fixed lead time of Model I is assumed to be the mean of the empirical lead time distribution used in Model III. We then compare both reorder points with actual observations during a simulated variable lead time. That is, for a particular item we simulate a random drawing from the lead time distribution, $g(\ell)$, shown in Figure 12. This simulated value is taken to be the length of the next lead time cycle. Observed demand is then determined for this lead time cycle and compared to the two predictions. Continuing this process for each item we can determine the cost and lead time supply effectiveness of the two models for an entire account. Indeed, we can compare the current Air Force methodology as well.

It should be clear as to why we compare the predictions to a simulated variable lead time. Although Model I makes predictions as if lead time was known and constant, its supply effectiveness, relative to Model III, must be measured relative to a realistic nonconstant lead time. After all, we only study the effectiveness of Model I in the hopes that it does provide reorder points that are reasonable in an actual environment of variable lead times. It would not be fair in judging the impact of variable lead time, to compare Model I's predictions to a constant lead time and Model III's to a variable lead time.

Table 19 shows some statistics on the cost and effectiveness of Model I, Model III and the current system for simulated lead time cycles. The items involved are ones from the Dover 59 account as resupplied from the S9E vendor (421 items). The costs are the dollar investments in reorder point spares; the percent sufficient shows how

Table 19. An Example of the Effect of Variable Lead Time

	Model I	Model III (G-P)	Current
Costs ($)	13,003.37	13,555.57	22,023.64
% Sufficient	89.3	89.5	91.2

effective the reorder point was in satisfying all demands during the simulated variable lead time period. For example, the reorder points for Model I would have satisfied all demands on 89.3% of the 421 items during the simulated variable lead time periods. There is practically no difference in the costs or percentages between Models I and III. This suggests that the impact of a variable lead time is virtually nonexistant, at least for this data set. This result was not unanticipated, given the similarity of Tables 12 and 18. It should be noted that either model does better than the current system--the percentages are reasonably close but Model I, say, has a reorder point spares investment which is 59.0% of the current system's cost.

For the purpose of sensitivity analysis, we are interested in how the reorder points might differ if the gamma distribution is used to approximate $g(\ell)$ in the numerical computation of $f(d)$. Although Figure 12 showed the inappropriateness of the gamma distribution as a probability model of lead time for the Dover data, it is important to determine if the reorder point computation is sensitive to the use of this approximation. Table 20 shows the reorder points (at .84) based on the gamma approximation. There are a few differences compared to Table 18 but whether or not they would appreciably alter the spares investment and lead time supply effectiveness will not be investigated. (An analysis like the one shown in Table 19 could be performed.) As discussed before, we think these present lead time data have too many probable anomalies to support a complete analysis at this time. Given that a "smooth" representation of the lead time data would most likely be required for implementation, the gamma approximation would be a reasonable one to consider in the future on better data. The few differences in Tables 18 and 20 suggest the gamma distribution would be a viable alternative to the inconvenience associated with tabulating each possible empirical distribution of lead time data.

Based on the above results (Tables 12, 18 and 19), we suspect that not all reorder point computations need to consider the effect of

Table 20. Reorder Points for Geometric-Poisson Probability Model (at 84% Confidence Level)

- Variable lead time
- Gamma approximation to Dover 59-S9E data

```
Variance/mean of lead time demand

5.0 ┤ 0  0  0  1  1  2  3  3  3  4  4  5  5  6  6  6
    │ 0  0  0  1  1  2  3  3  3  4  4  5  5  5  6  6
    │ 0  0  0  1  2  2  3  3  3  4  4  5  5  5  6  6
    │ 0  0  0  1  2  2  3  3  3  4  4  5  5  5  6  6
    │ 0  0  0  1  2  2  3  3  3  4  4  5  5  5  6  6
4.0 ┤ 0  0  0  1  2  2  3  3  3  4  4  5  5  5  6  6
    │ 0  0  0  1  2  2  3  3  3  4  4  5  5  5  6  6
    │ 0  0  0  1  2  2  3  3  3  4  4  5  5  5  6  6
    │ 0  0  0  1  2  2  3  3  3  4  4  4  5  5  6  6
    │ 0  0  1  1  2  2  3  3  3  4  4  4  5  5  5  6
3.0 ┤ 0  0  1  1  2  2  3  3  3  4  4  4  5  5  5  6
    │ 0  0  1  1  2  2  3  3  3  4  4  4  5  5  5  6
    │ 0  0  1  1  2  2  2  3  3  4  4  4  5  5  5  6
    │ 0  0  1  1  2  2  2  3  3  4  4  4  5  5  5  6
    │ 0  0  1  1  2  2  2  3  3  3  4  4  4  5  5  5
2.0 ┤ 0  0  1  1  2  2  2  3  3  3  4  4  4  5  5  5
    │ 0  0  1  1  2  2  2  3  3  3  4  4  4  5  5  5
    │ 0  0  1  1  2  2  2  3  3  3  4  4  4  5  5  5
    │ 0  0  1  1  2  2  2  3  3  3  4  4  4  4  5  5
    │ 0  0  1  1  2  2  2  3  3  3  3  4  4  4  5  5
1.0 ┤ 0  0  1  1  1  2  2  2  3  3  3  4  4  4  5  5
    └──┴─────┴─────┴─────┴─────┴─────┴─────┴─────┴──
      .1    .5    .9    1.3   1.7   2.1   2.5   2.9

              Mean lead time demand
```

variable lead time. Indeed, we suspect for CONUS account-resupplier combinations that have reasonably symmetric lead time distributions, the effect of variable lead time can be ignored. Such a supposition for overseas bases, as resupplied by CONUS depots, is not so obvious, though. In fact, the larger VMR of lead time (Table 16) for the Bentwaters accounts will more directly influence the VMR of demand during a variable lead time, $VMR_{(var)}$, in equation (7). All of the other variables in equation (7) are approximately the same for the Bentwaters and CONUS accounts (Tables 5 and 6), so we would expect the larger $VMR_{(L)}$ for Bentwaters to produce a larger $VMR_{(var)}$, and in turn affect the reorder point computation. Table 21 shows a comparison of the reorder point tables for the Bentwaters 16 account. Both tables

Table 21. A Comparison of Reorder Points for Fixed and Variable
Lead Time (Geometric-Poisson and Bentwaters 16-FGZ)

(fixed)

Variance/mean of lead time demand

5.0	0	2	3	4	5	6	6	7	7	8	8	9	9	10	10	11
	0	2	3	4	5	5	6	7	7	8	8	9	9	10	10	10
	0	2	3	4	5	5	6	7	7	8	8	9	9	9	10	10
	0	2	3	4	5	5	6	6	7	7	8	8	9	9	10	10
	0	2	3	4	5	5	6	6	7	7	8	8	9	9	9	10
4.0	0	2	3	4	4	5	6	6	7	7	8	8	8	9	9	10
	0	2	3	4	4	5	5	6	6	7	7	8	8	9	9	9
	0	2	3	4	4	5	5	6	6	7	7	8	8	8	9	9
	0	2	3	4	4	5	5	6	6	7	7	7	8	8	9	9
	0	2	3	3	4	5	5	6	6	6	7	7	8	8	8	9
3.0	0	2	3	3	4	4	5	5	6	6	7	7	7	8	8	9
	0	2	3	3	4	4	5	5	6	6	6	7	7	8	8	8
	0	2	2	3	4	4	5	5	5	6	6	7	7	7	8	8
	0	2	2	3	4	4	4	5	5	6	6	6	7	7	8	8
	0	2	2	3	3	4	4	5	5	5	6	6	7	7	7	8
2.0	0	2	2	3	3	4	4	5	5	5	6	6	6	7	7	7
	0	1	2	3	3	3	4	4	5	5	5	6	6	6	7	7
	0	1	2	2	3	3	4	4	4	5	5	6	6	6	6	7
	0	1	2	2	3	3	3	4	4	5	5	5	6	6	6	7
	0	1	2	2	3	3	3	4	4	4	5	5	5	6	6	6
1.0	0	1	2	2	2	3	3	3	4	4	4	5	5	5	6	6
	.1		.5		.9		1.3		1.7		2.1		2.5		2.9	

Mean lead time demand

(variable)

Variance/mean of lead time demand

5.0	0	2	3	4	5	6	7	7	8	8	9	10	10	11	11	12
	0	2	3	4	5	6	6	7	8	8	9	9	10	11	11	12
	0	2	3	4	5	6	6	7	8	8	9	9	10	10	11	11
	0	2	3	4	5	6	6	7	7	8	9	9	10	10	11	11
	0	2	3	4	5	5	6	7	7	8	8	9	10	10	11	11
4.0	0	2	3	4	5	5	6	7	7	8	8	9	9	10	10	11
	0	2	3	4	5	5	6	6	7	8	8	9	9	10	10	11
	0	2	3	4	4	5	6	6	7	7	8	9	9	10	10	11
	0	2	3	4	4	5	6	6	7	7	8	8	9	9	10	10
	0	2	3	4	4	5	5	6	7	7	8	8	9	9	10	10
3.0	0	2	3	3	4	5	5	6	6	7	8	8	9	9	10	10
	0	2	3	3	4	5	5	6	6	7	7	8	8	9	9	10
	0	2	3	3	4	4	5	6	6	7	7	8	8	9	9	10
	0	2	2	3	4	4	5	5	6	6	7	7	8	8	9	9
	0	2	2	3	4	4	5	5	6	6	7	7	8	8	9	9
2.0	0	2	2	3	4	4	5	5	6	6	7	7	8	8	9	9
	0	1	2	3	3	4	4	5	5	6	6	7	7	8	8	9
	0	1	2	3	3	4	4	5	5	6	6	7	7	8	8	9
	0	1	2	3	3	4	4	5	5	6	6	6	7	7	8	8
	0	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8
1.0	1	1	2	2	3	3	4	4	5	5	6	6	6	7	7	8
	.1		.5		.9		1.3		1.7		2.1		2.5		2.9	

Mean lead time demand

51

are based upon a 0.95 probability (the convention for overseas bases).
The top table gives the reorder points for a fixed lead time cycle.
The lower table gives the reorder points for a variable lead time
distribution. Here the Bentwaters 16-FGZ empirical data were used.
Clearly, more of the reorder points differ among the two tables than
was evident in the Dover 59-S9E analysis. It is reasonable to expect,
then, that the influence of a variable lead time will be greater for
the overseas bases than the CONUS bases.

The effect of variable lead time could also be studied for the
constant-Poisson distribution in the same way. We would take $f(d|\ell)$
as the constant-Poisson and use $g(\ell)$ to form the marginal distribution
of demand during lead time, $f(d)$, as in equation (4). No data will be
shown here; the results parallel those for the geometric-Poisson model.
That is, marginal influence for the CONUS accounts and substantially
more influence for the Bentwaters account.

An important advantage for the gamma approximation would result in
the constant-Poisson case. If we could assume that lead time had
approximately a gamma distribution, then $N(L)$ in equation (3) would have
a negative binomial distribution [8]. In essence, the number of customer
arrivals during a variable lead time would be negative binomial and for
constant demands per customer, $D(L)$ in equation (3) would have a negative
binomial distribution defined on the integers $0, c, 2c, \ldots$ These
probabilities could be computed analytically as opposed to numerically
or iteratively and thus machine time for implementation would be
considerably reduced.

In this section we have attempted to suggest the impact of variable
lead time on the reorder point computation. By necessity, due to some
probable anomalies in the lead time data, only cursory analyses were
shown. A methodology was presented that can be used in the future to
examine more carefully the effect of variable lead time. We suspect
that variable lead time will be inconsequential for CONUS bases but
will be significant for overseas bases.

SECTION 6
RECOMMENDATIONS FOR VALIDATION/IMPLEMENTATION

Now that several models have been presented and subjected to data, we comment on some items that are important for further validation testing and possible implementation.

By far the strongest comment we have is that a computer implementation of any model must provide the capability for identifying anomalous data. The lead time data we investigated were certainly suspicious and some of the data on customer arrivals and demands were also. Parameter estimates for model inputs are often seriously affected by atypical data; more importantly, though, is that predictions based on the model may not be realistic.

The Federal Simulation (FEDSIM) model at the AFLMC provides the capability for another check on the work reported here. Some comments about the use of FEDSIM are appropriate at this point. First, the stockage effectiveness (for example, 0.84 for CONUS bases) specified as a target in the current system clearly relates to the lead time cycle only. We suspect that this is not the common interpretation among supply managers. In fact, we doubt that the FEDSIM model computes the effectiveness during the lead time cycle. To substantiate the work reported here, we recommend that the stockage effectiveness computation described above be added to FEDSIM.

Second, careful thought should be given to the length of the base period that would be used in FEDSIM to compare the current system and the models presented in this report. We would certainly not recommend an arbitrary choice. The models we presented are probabilistic models with assumed constant parameters. As pointed out before, periods longer than 9-12 months probably would tend to violate stationarity assumptions. It seems reasonable that a prediction for a lead time of 3-7 weeks should be possible with historical data limited to 9-12 months.

Third, the comparisons reported here for the current system versus the probabilistic models did not use any weighted schemes, such as

53

exponential smoothing, to forecast parameter values. Whether or not weighted schemes should be used in FEDSIM comparison runs needs to be addressed.

A summary comment about FEDSIM use is that standard experimental design techniques should be considered in setting up comparison runs. It would be very easy to confound or mask any true differences without an objective test plan.

The zero demand items were ignored in our analysis but clearly a technique is needed to set reorder points on them. Given the result in Table 12 where items with a very low forecasted mean lead time demand have a reorder point of zero, we submit that zero demand items should have a zero reorder point as well. If the reorder point for an item with, say, two demands for a 26 week observation period is zero (its mean lead time demand would be approximately 0.2 for a lead time of three weeks), then surely an item with no demands should be zero also. Further testing of this idea could be performed with the FEDSIM model.

Another result that could be investigated easily with the FEDSIM model is the effect of variable lead time. Just as was performed in this report, a reorder point could be compared with observed demands in a fictitious lead time period. In one case the fictitious lead time period would be equal to the mean lead time and in another case, the lead time would be a random drawing from the empirical lead time distribution. If the reorder point sufficiency was approximately the same in both cases, then the effect of variable lead time could be considered neglible.

As a last remark, we recommend that a full base (all FSGs) be used in comparison runs as described in this report. Coupled with the broad bases and FSGs already studied here, the results for an entire base would form an in-depth package for evaluation.

APPENDIX A

FORMULATION OF TOTAL VARIABLE COST

For a particular stock item the total variable cost (TVC) per year can be expressed as

$$TVC = OC + HC + BC \qquad (A-1)$$

where: OC = total annual order costs
HC = total annual holding costs
BC = total annual backorder costs.

The order costs can be quite easily calculated as

$$OC = \frac{D}{Q} A \qquad (A-2)$$

where: D = mean annual demand rate
Q = order quantity
A = cost per order.

In this form the order costs are consistent with the typical approach to determining the total costs in an inventory model [7]. The holding costs are governed by the relationship

$$HC = (R + \frac{Q}{2} - DL)IC \qquad (A-3)$$

where: R = the reorder point
L = the mean lead time in years
I = holding cost rate
C = unit price.

This formulation of the holding costs assumes that the expected number of backorders is negligible and calculates the expected on hand inventory as the net inventory. It should be noted that this formulation is consistent with the typical inventory model.

The third cost term, annual backorder costs, can be determined by a variety of methods. DODI 4140.45 proposes a method based on the quantity "Time Weighted Requisitions Short." This formulation appears to have Air Force acceptance and will be used in this discussion; thus,

$$BC = \lambda(TWRS) = \frac{\lambda E}{SQ} \int_R^\infty (x-R)[F(x+Q;L) - F(x;L)]dx \qquad (A-4)$$

with $\lambda(TWRS)$ = the implied penalty cost of time weighted requisitions short

where: λ = shortage cost parameter
        E = item essentiality
        S = average units per requisition
        X = random variable for demands during constant lead time L
$F(.;L)$ = cumulative probability distribution of demand during
            lead time L .

As pointed out in [15] for many practical applications, the expected number of units short

$$\frac{1}{Q} \int_{R}^{\infty} (x-R)[F(x+Q;L) - F(x;L)]dx \qquad (A-5)$$

can be approximated by

$$\frac{1}{Q} \int_{R}^{\infty} (x-R)f(x;L)dx \qquad (A-6)$$

where $f(x;L)$ is the probability distribution of demand during lead time. Using this approximation the final term in the cost equation is

$$\lambda (TWRS) = \frac{\lambda E}{SQ} \int_{R}^{\infty} (x-R)f(x;L)dx . \qquad (A-7)$$

We have now arrived at the total variable cost per year per item:

$$TVC = \frac{D}{Q} A + (R + \frac{Q}{2} - DL)IC + \frac{\lambda E}{SQ} \int_{R}^{\infty} (x-R)f(x;L)dx . \qquad (A-8)$$

To determine the optimal operating doctrine in terms of order quantity, Q, and the reorder point, R, we must calculate the expressions

$$\frac{\partial (TVC)}{\partial Q} = \frac{\partial (TVC)}{\partial R} = 0 . \qquad (A-9)$$

Doing this, we obtain the simultaneous equations

$$IC = \frac{\lambda E}{SQ} \int_{R}^{\infty} f(x;L)dx \qquad (A-10)$$

$$Q = \sqrt{\frac{2[DA + \frac{\lambda E}{S} \int_{R}^{\infty} (x-R)f(x;L)dx]}{IC}} . \qquad (A-11)$$

In general, an easy and quick solution to these equations does not exist. Hadley and Whitin [7] suggest an iterative procedure for solving for R and Q. Even though this procedure is sound analytically, the iterative technique would have to be accomplished on every distinct item in the stock account.

The framers of the policy contained in DODI 4140.45 obviously recognized the computation and time involved in determining the optimal Q and R via the exact formulation above. To achieve a balance between near optimal solutions and computational effort the DODI authorizes the use of an approximate model. In this model the optimal Q and R are determined by a two-step process. The first step assumes there are no backorders and the related total variable cost per year is the sum of the holding and ordering costs, thus,

$$TVC = \frac{D}{Q} A + (R + \frac{Q}{2} - DL) IC . \tag{A-12}$$

Solving this equation for the optimal Q yields the standard Wilson EOQ equation

$$Q = \sqrt{\frac{2DA}{IC}} . \tag{A-13}$$

The difference between this order quantity and the optimal obtained from the exact formulation is an additional term to account for the stochastic nature of demands by including the expected number of backorders (requisitions short). If the expected number of requisitions short is small, then the Q obtained from the Wilson EOQ equation yields a good approximation to the optimal.

The second step of the approximation process is used to determine the reorder point, R. The variable cost is assumed to consist of a holding cost and a shortage cost or

$$VC = IC \int_0^R (R-x) f(x;L) dx + \frac{\lambda}{S} \int_R^\infty (x-R) f(x;L) dx . \tag{A-14}$$

Solving this expression for the optimal R yields the equation

$$\frac{SIC}{SIC + \lambda} = \int_R^\infty f(x;L) dx . \tag{A-15}$$

At this point the analysis breaks down because the underlying probability distribution, $f(x;L)$, is unknown. The priority item in this research is to determine the distribution. The DODI suggests using an equation of the form

$$R = DL + t\sigma \tag{A-16}$$

where: σ = standard deviation of lead time demand
       t = safety level parameter.

   The Air Force is currently using a reorder point formula of this form, with σ being approximated by the square root of three times the expected demand during lead time and t = 1 for CONUS. Additionally, if the demand during lead time is normally distributed, the value for t yields a .16 probability of a stockout.

REFERENCES

1. Boswell, M.T. and G.P. Patil, "Chance Mechanisms Generating the
   Negative Binomial Distribution," published in Random Counts in
   Models and Structures, Vol 1, edited by G.P. Patil, Pennsylvania
   State University Press, 3-22 (1970).

2. Rott, D.L., "Compound Distributions with Efficient Computation in
   Inventory Model Applications," Naval Research Logistics
   Quarterly, 24, No. 3, 407-416 (1977).

3. Brown, B.B. and M.A. Geisler, Analysis of the Demand Patterns for
   B-47 Airframe Parts at Air Base Level, the RAND Corp, RM-1297,
   1954.

4. Department of Defense Instruction 4140.45, Standard Stockage Policy
   for Consumable Secondary Items at the Intermediate and Consumer
   Levels of Inventory, Assistant Secretary of Defense (Manpower,
   Reserve Affairs, and Logistics), April 7, 1978.

5. Feller, W., An Introduction to Probability Theory and Its
   Applications, 1, John Wiley and Sons, Inc., New York, 1967.

6. Galliher, H.P., P.M. Morse, and M. Simond,"Dynamics of Two Classes
   of Continuous Review Inventory Systems," Operations Research, 1,
   No. 3, 362-384 (1959).

7. Hadley, G. and T.M. Whitin, Analysis of Inventory Systems, Prentice-
   Hall, Inc., Englewood Cliffs, New York, 1963.

8. Johnson, N.L. and S. Kotz, Discrete Distributions, Houghton Mifflin,
   Boston, 1969.

9. Kendall, M.G. and A. Stuart, Advanced Theory of Statistics, Vol II,
   Hafner Publishing Co., New York, 1973.

10. Kruskal, W.H. and J.M. Tanur, eds., International Encyclopedia of
    Statistics, Vol 1, McMillan and Free Press, New York, p. 24, 1978.

11. Mathematics of Military Action, Operations and Systems, Department
    of Army Pamphlet, No. 70-5, Headquarters, Dept of the Army, 1968.

12. McFadden, F.R., "On Lead Time Demand Distributions," Decision Sciences,
    3, 106-126 (1972).

13. Meyer, P.L., Introductory Probability and Statistical Applications,
    2nd ed., Addison Wesley Publishing Co., Reading, MA, 1970.

14. Mitchell, C.R. and S.J. Monaco, "The Unbiquity of the Negative Binomial Distribution in Aircraft Logistics," Dept of Mathematical Sciences Technical Note, USAF Academy, 1979.

15. Orr, D. and A.J. Kaplan, The Economic Stockage Model, Inventory Research Office, Institute for Logistics Research, US Army Logistics Management Center, 1971.

16. Recommendation on Implementation of Massachusetts Institute of Technology Research in Secondary Item Supply Control, Ordnance Management Project, Office of Ordnance Research, AD 268 372, 1959.

17. Seppanen, M.S., Technical Planning for the USAF Standard Base Supply System (SBSS); RIMSTOP and Beyond, Department of Industrial Engineering, The University of Alabama, 1978.

18. Sherbrooke, C.C., Discrete Compound Poisson Processes and Tables of the Geometric Poisson Distribution, Naval Research Logistics Quarterly, 15, No. 2, 189-204 (1968).

19. Ward, J.B., "Determining Reorder Points When Demand is Lumpy," Management Science, 24, No. 6, 623-632 (1978).